MCG3340 — Fluid Mechanics I

Lecture Notes

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1 Introduction

This chapter serves to give specific definitions for concepts that will be important throughout this course.

1.1 What is fluid mechanics?

Before beginning a study of *fluid mechanics*, it is obviously a good idea to have a firm definition of what this term means. It is expected that the term *mechanics* should be quite familiar. It is the study of motions and deformations of bodies as well as the forces that cause and result from these displacements and deformations.

1.2 What is a fluid?

If a definition of mechanics is assumed to be familiar, it may seem that the definition of a *fluid* must be much more easily obtained. However, this is deceptively untrue. Though we are all familiar with "classical" examples of fluids, for example, **liquids** and **gases**, what properties do these states have that make them *fluid*? Several definitions that are commonly given are:

1. Liquids, gases, and plasmas (ionized gases),
2. Any material that deforms continuously under the action of a shear stress,
3. Anything that takes the form of its container,
4. Anything that "flows".

In my opinion, none of these definitions is ideal. Definition 1 is simply a list of examples of fluids. It does not really specify exactly what it is about these states that makes them a fluid. Definition 2 is very common in engineering textbooks. For the vast majority of practical situation, it is true. Solids subjected to a shear stress can come to an equilibrium, whereas a fluid in thermodynamic equilibrium* subjected to a shear stress cannot be in a stationary equilibrium. However, this is not necessarily true for very extreme situations. For example, for highly rarefied gas flows, it is possible to have a gas experience a shear stress and yet have no bulk motion [2]. However practical, this is therefore also not a perfect definition. Additionally, it does not explain *why* fluids normally display this behaviour. Definition 3 also gives a property that all fluids have, but does not explain why a fluid has this property. It is also ambiguous as to whether things like sand or sugar, which takes the shape of their container, are fluids. Definition 4 is very similar to Definition 3. Is sand flowing down an incline or in an hourglass an example of a fluid?

The definition of a fluid that I prefer is:

Fluid

A material that is made up of a huge number of constituent particles, all of which are in continuous, seemingly random motion relative to each other.

Obviously, liquids and gases fall under this definition. The atoms and molecules of mater in these states are continuously in "random" motion relative to one another. Any two particles that are neighbours at a given instant will quickly move apart and will likely not interact again for an extremely long time (if ever). Solids do not fall into this definition because the molecular structure of solids holds atoms in an arrangement that is largely fixed for reasonable times.

According to the definition that I prefer, sand is a fluid a times and not a fluid at other times. While it is being poured, the individual grains of sand are in constant relative motion—it is a fluid. However, when it comes to rest, I would say that it is no longer a fluid. Other situations that I would consider "fluids" include avalanches, fluidized-bed boilers and reactors, as well as the movement of interstellar dust in the galaxy. The techniques commonly used to treat these situations are exactly the same as the techniques we will learn to apply to more traditional fluids such as liquids and gases.

1.3 How many values are needed to describe a fluid's state?

The most direct way to describe a fluid would be to give a full description of each of its constituent particles. This would provide a complete description of the fluid. Unfortunately, such a direct description is far beyond the scope of even the world's larges computers. At standard atmospheric conditions, $1\,\mathrm{m}^3$ of air contains on the order of 10^{25} particles—an unimaginably huge number. A direct description down to the smallest scales is therefore impossible. Fortunately, the number of variables needed for the large majority of practical situations is far less. A thorough proof of why this is the case is beyond this course. However, a simplified explanation is possible.

*The concept of thermodynamic equilibrium should be familiar from your studies of thermodynamics. A material is in thermodynamic equilibrium if it is only subjected to processes that happen "slowly enough". A more convincing definition is given in Section 1.4.

Table 1.1: Probability of not getting 50% heads, plus or minus 1%, for N tosses of a fair coin.

Number of tosses of a fair coin, N	chances of *not* getting $50\,\% \pm 1\,\%$ heads
10	75.4 %
100	76.4 %
1000	50.7 %
10 000	4.44 %
100 000	2.48×10^{-8} %
1 000 000	5.33×10^{-87} %
2 000 000	5.15×10^{-174} %
3 000 000	5.74×10^{-261} %

We are very lucky that, for the vast majority of situations, an argument of probability allows us to describe the state of a fluid using only a small number parameters. The best way to explain this is to begin with a simpler situation. Imagine, for a moment, a coin toss game where one player tosses a coin in the air some number of times and the percentage of "heads" and "tails" is recorded. A second player must guess what percentage of the time the coin landed "heads up". To win, the second player must be correct to within 1 %. A smart player would always guess 50 % heads and 50 % tails, as this is always the most likely outcome. However, the chances this smart player would win or lose still depends on the number of times the coin is tossed. If the coin is tossed ten times, there is a 75.4 % chance that the coin would not land on heads five times. It the coin is tossed one hundred times, there is a 76.4 % chance that the coin will not land on heads forty-nine, fifty, or fifty-one times. As the number of tosses increases, the chances of the smart player loosing drop very quickly. This is shown in Table 1.1. For a million coin tosses, there is only a 5.33×10^{-87} % chance that the number of heads won't be within one percent of the most likely outcome. It is basically a sure thing.

Certainly the state of a fluid particle is not binary (heads or tails). It can have any velocity and any position in a container. However, for an ideal gas, it can be shown that there is a "most likely" state for the gas and, because of the enormous number of particles in any typical fluid, that the chances of deviation from this state are practically zero [1]. Though a firm proof of the existence of such a statie is only available for ideal gases, it is generally believed that this concept carries over to all matter.

The only contrains of this most likely state is that it cannot violate the conservation laws (mass, momentum, and energy). Since the most likely state of a fluid is only a function of these three, independent, conserved things, and it is a "sure thing" that it is in this state, it seems that **two independent scalars and one vector should be enough to characterize the state of a fluid**.

1.4 The continuum approximation

If one accepts the definition that a fluid is a collection of an immensely large number of similar particles that are in constant relative motion, the question arises "How much information is needed to describe the properties of such a collection?" Even in the simple case that each particle is completely characterized by its position and velocity, a complete description of the fluid would require the knowledge of six variables per particle (x, y, and z position and velocity coordinates). This is obviously impossible. Fortunately, the huge separation of spacial scales between the typical lengths related to the particle nature of a fluid and the typical lengths of practical situations allow us to ignore the particles to a certain extent.

Just as the individual discrete pixles of a digital image become indistinguishable from continuous variations of colour as one zooms out, on a large scale we are entirely justified in approximating fluid properties using continuous fields. This is known as the ***conitnuum approximation***.

The continuum approximation

Though a fluid is comprised of particles, on a large scale this particle nature is unimportant. The fluid can be well described by continuous fields.

1.5 Compressible vs. incompressible fluids

In fluid mechanics we often characterize a fluid as being either compressible or incompressible. The definition of an incompressible fluid is any fluid whose mass density, ρ, cannot change. In reality, no fluid is perfectly incompressible and small changes in density are always present. However, liquids tend to be very nearly incompressible. For this reason, for most applications, it is a very good approximation to treat liquids as having a constant density—incompressible. Gases, on the other hand, can undergo density changes due to changes in temperature or pressure, and are thus compressible.

Throughout this course, we will see that the analysis of fluid behaviour is often greatly simplified if the mass density is taken as a constant. For many specific situations, even compressible fluids do not undergo significant density changes. For these situations, even gases, which can be compressed given the proper conditions, can be treated as incompressible.

2 Fluid Statics

Fluid statics is the study of fluid that are at rest in a particular coordinate system. By "at rest", we mean that the average velocity of the particles making up the fluid around any location is zero. The techniques developed in this chapter are applicable to fluids that are at rest, or at rest in an inertial coordinate system*. Perhaps surprisingly, fluid undergoing certain accelerations can also be treated by the techniques of this chapter, as they appear at rest in particular types of non-inertial (accelerating) frames.

As we saw in Section 1.3, we assume that we need five variables to describe the state of a fluid. Three of these values come from the conservation of the three components of momentum. For fluid statics, the fluid appears to be at rest, and thus it is assumed that we already have three pieces of information (all components of the fluid's velocity are zero). We therefore need to find relations between the remaining two state parameters. As we will see, it is often convenient to take these parameters to be the fluid's mass density, ρ, and its pressure, p.

2.1 What do we mean by fluid pressure?

The concept of pressure is very important in the field of fluid mechanics. We often speak of the pressure of a fluid in a closed vessel, for example, your bicycle tire may have a fluid pressure of 60 psi. Alternatively, you may hear that the day's atmospheric pressure is 100.678 kPa. What, exactly, does this mean about the state of the fluid?

To start, it's a good idea to remember what the definition of pressure is. When two bodies on in contact and exert a compressive force on each other, the pressure is the ratio between this force and the area of contact,

$$p = \frac{|\vec{F}_p|}{S}. \tag{2.1}$$

If the force is not acting evenly over the whole surface, one can break the surface into differential elements that are small enough that the force is evenly distributed over the small region. the

*An inertial coordinate system is a coordinate system is a coordinate system that is in only constant rectilinear motion (no accelerations). Classical laws of physics, such as Newton's laws of motion, take their simplest form in an inertial frame of reference.

local pressure on this tiny element is then the ratio of the small force exerted on the element and the element area.

Pressure

The ratio between the magnitude of the compressive normal force of contact between two bodies, $\mathrm{d}\vec{F}_p$, and the area of application, $\mathrm{d}S$,

$$p = \frac{|\mathrm{d}\vec{F}_p|}{\mathrm{d}S}. \tag{2.2}$$

Though the term is very similar, there is an important difference between the concept of *pressure* and the concept of *fluid pressure*. For starters, we speak of a fluid having a pressure far away from solid boundaries on which it could exert a force. If the fluid pressure at some point in the atmosphere is 101 325 Pa, does that mean that somehow there is a force being applied to an invisible surface? If there is a force, in what direction does it act? The answer to these questions is not clear. The only way to measure such a force would be with a solid meter of some kind. The fluid would then be exerting a force on the real solid boundary of the measuring device.

The definition of fluid pressure that we will use is the ratio between the force that would be exerted by a fluid and the differential area, $\mathrm{d}S$, of a solid boundary *if* a tiny solid boundary were introduced at that point in the fluid. Experience shows that this resulting force would be normal to the solid boundary and compressive. The equation describing the force that would be exerted on this differential surface is therefore

$$\mathrm{d}\vec{F}_p = -p\,\hat{n}\,\mathrm{d}S\,,$$

where $\hat{n}$ is the unit vector that is normal to the differential surface and points into the fluid. The negative sign means that the force will be onto the surface (compressive).

Though the resulting force is a vector with a direction, it is very important to remember that, using this definition, the fluid pressure is a scalar quantity. It has no direction and the direction of the resulting force is entirely determined by the orientation of the immersed surface.

Fluid pressure

$$\mathrm{d}\vec{F}_p = -p\,\hat{n}\,\mathrm{d}S \tag{2.3}$$

2.2 Hydrostatic forces on submerged surfaces

To find the total resultant pressure force on a solid surface, S, we must add up the contribution of the force generated on each individual differential surface element. That is, we must integrate Equation (2.3), over the entire surface,

$$\vec{F}_p = \iint_S -p\hat{n}\,\mathrm{d}S\,. \tag{2.4}$$

In this equation, both p and $\hat{n}$ can vary over the surface and are, generally, a function of position.

2.3 Variation of fluid pressure in a fluid at rest

In order to determine how the fluid pressure varies with location in a fluid that is at rest, we will consider a stationary, closed container filled with a stationary fluid that is in equilibrium. We will consider the case when there are only two forces acting on the fluid: the weight of the fluid and the pressure force at the wall. As the system is assumed to be in equilibrium, these two forces must sum to zero.

The weight of the fluid, $\vec{F}_g$, is simply the integral of the mass density times gravitational acceleration, $\vec{g}$, over the volume of the container,

$$\vec{F}_g = \iiint\limits_V \rho \vec{g} \, \mathrm{d}V \,. \tag{2.5}$$

This force must be balanced with the net pressure force exerted by the container walls on the fluid, $\vec{F}_p$. Equation (2.4) give us the relation for the force exerted by the fluid on the walls of the container. Remember, in this expression, the unit normal, $\hat{n}_{\text{in}}$, must be taken as pointing into the fluid,

$$\vec{F}_p = \oint\!\!\!\oint\limits_S -p \, \hat{n}_{\text{in}} \, \mathrm{d}S \,. \tag{2.6}$$

If this is the force exerted by the fluid on the walls of the container, the container must exert a force of equal magnitude but opposite direction on the fluid. We get this force by switching the direction of the unit normal. The pressure force exerted by the walls on the fluid is therefore

$$\oint\!\!\!\oint\limits_S -p \, \hat{n}_{\text{out}} \, \mathrm{d}S \,. \tag{2.7}$$

The force balance is therefore

$$\iiint\limits_V \rho \vec{g} \, \mathrm{d}V - \oint\!\!\!\oint\limits_S p \, \hat{n}_{\text{out}} \, \mathrm{d}S = 0 \,. \tag{2.8}$$

We can now use the gradient theorem† on the second term to write this as

$$\iiint\limits_V \rho \vec{g} \, \mathrm{d}V - \iiint\limits_V \vec{\nabla} p \, \mathrm{d}V = 0 \,. \tag{2.9}$$

As the two integrals are over the same volume, they can be combined as

$$\iiint\limits_V \left(\rho \vec{g} - \vec{\nabla} p \right) \mathrm{d}V = 0 \,. \tag{2.10}$$

†If you are not comfortable with the gradient theorem, see Section A.7 of Appendix A.

The shape of the container, and thus the volume of integration, is arbitrary. The only way that this integral will be equal to zero, regardless of the shape and size of V, is if the integrand is always equal to zero,

$$\rho\vec{g} - \vec{\nabla} p = 0 . \tag{2.11}$$

This leads to the final expression for the variation of pressure in a fluid at rest.

Variation of pressure in a fluid at rest

In a fluid that is at rest, subject only to a gravitational field, the pressure varies according to the law

$$\vec{\nabla} p = \rho\vec{g} . \tag{2.12}$$

2.3.1 Fluid pressure in a fluid at rest is a function of depth only

If we assume that we are working in a Cartesian coordinate system with the z axis in the upward directions, we have

$$\vec{g} = \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -9.81\,\mathrm{m/s^2} \end{bmatrix} \tag{2.13}$$

We therefore have,

$$\frac{\partial p}{\partial x} = 0 , \qquad \frac{\partial p}{\partial y} = 0 , \qquad \frac{\partial p}{\partial z} = -\rho g . \tag{2.14}$$

As the derivative of pressure in both the x and y directions are zero, the pressure does not change in these directions. Another way of putting this is that the pressure in a fluid at rest is only a function of the depth. The variation of the pressure is described as the depth changes is described by the equation

$$\frac{\mathrm{d}p}{\mathrm{d}z} = -\rho g . \tag{2.15}$$

Example 2.1 : Variation of air pressure in a mine

2.3.2 Relation between the fluid pressure and fluid weight

One way to understand what gives rise to the change in fluid pressure in a fluid that is at rest is to see the relationship between the a fluid's weight and its pressure. To do this, imagine a vertical cylinder filled with a fluid at rest. To maintain this equilibrium, all forces acting on the fluid must sum to zero. Again, there are only two forces acting on the fluid: gravity and pressure forces,

$$\sum \vec{F} = \vec{F}_g + \vec{F}_p = 0 . \tag{2.16}$$

The fluid weight is the integral of the fluid's mass density over the volume. It acts downwards in the vertical direction. The net pressure force on the fluid is the integral of the differential pressure force, described by the negative of Equation (2.3)[‡] , over the surface of the cylinder. The pressure force exerted by any part of the vertical walls of the cylinder on the fluid can only have a horizontal component. The pressure force exerted by the top and bottom of the cylinder will produce forces that are uniquely in the vertical direction. We will look at the vertical component of Equation (2.16) and will take upward to be the positive direction. This gives

$$\iiint_V \rho g_z \mathrm{d}V + \iint_{S_{\text{top}}} p_{\text{top}} \left(\hat{n}_{\text{top}} \cdot \hat{k} \right) \mathrm{d}S + \iint_{S_{\text{bottom}}} p_{\text{bottom}} \left(\hat{n}_{\text{bottom}} \cdot \hat{k} \right) \mathrm{d}S = 0, \tag{2.17}$$

with $g_z = -9.81\,\mathrm{m/s^2}$. I this situation, $\hat{n}_{\text{top}}$ is constant and points down (into the fluid), while $\hat{n}_{\text{bottom}}$ is constant and points up. As the pressure is only a function of the vertical height, it is also constant on the top and bottom of the cylinder. This leads to expression

$$\iiint_V \rho g_z \mathrm{d}V + \iint_S \left(p_{\text{bottom}} - p_{\text{top}} \right) \mathrm{d}S = 0$$

$$\iiint_V \rho g_z \mathrm{d}V + \left(p_{\text{bottom}} - p_{\text{top}} \right) S = 0. \tag{2.18}$$

This shows that the change in pressure in a fluid is caused by the weight of the fluid. As we descend, the pressure must increase exactly enough to account for the weight of the fluid that is now above.

2.3.3 Variation of pressure in an incompressible fluid at rest

In general, the fluid density can be a function of the fluid pressure (and thus position). Therefore, this relation for density as a function of pressure must be inserted into Equation (2.12 before integration. There is one situation, however, when this is greatly simplified: that of an incompressible fluid.

We remember that, for an incompressible fluid, the density is constant. For many practical situations, the gravitational acceleration is also a constant vector. Taking the z direction to be vertical up, Equation (2.12) becomes

$$\frac{\mathrm{d}p}{\mathrm{d}z} = \rho g, \tag{2.19}$$

with $g = -9.81\,\mathrm{m/s^2}$. This relation can be integrated easily between two points,

$$\int_{p_1}^{p_2} \mathrm{d}p = \rho g \int_{z_1}^{z_2} \mathrm{d}z. \tag{2.20}$$

[‡]Equation (2.3 give the force exerted by the fluid on the wall, we are now looking at the force that the wall exerts on the fluid.

This leads to

$$\begin{aligned} p_2 - p_1 &= \rho g \left(z_2 - z_1\right) \\ \Delta p &= \rho g \Delta z\,. \end{aligned} \tag{2.21}$$

Of course, this simple relation depends on the assumption that both ρ and $\vec{g}$ are constant. Thus, for incompressible fluids in a constant gravitational field, this simplified form may be used.

2.3.4 Absolute and relative pressure

In many situations, it is not the absolute (real) pressure of a fluid that is most convenient. Often, the difference in pressure between the absolute pressure at some location and the atmospheric pressure is of interest. This is difference is defines as the relative pressure.

Relative pressure

The relative pressure, p_{rel}, is the difference between the absolute pressure and the atmospheric pressure,

$$p_{\text{rel}} = p_{\text{abs}} - p_{\text{atm}} \tag{2.22}$$

The relative pressure is primarily important when we are computing forces on surfaces with the atmosphere is on one side. The magnitude of the net pressure force on each differential surface element will be equal to the absolute pressure on one side minus the atmospheric pressure on the other side, times the differential surface size. That is,

$$\begin{aligned} \mathrm{d}\vec{F}_{\text{net}} &= -p_{\text{abs}}\,\hat{n}\,\mathrm{d}S + p_{\text{atm}}\,\hat{n}\,\mathrm{d}S \\ &= -\left(p_{\text{abs}} - p_{\text{atm}}\right)\,\hat{n}\,\mathrm{d}S \\ &= -p_{\text{rel}}\,\hat{n}\,\mathrm{d}S\,. \end{aligned} \tag{2.23}$$

2.4 Moments generated on submerged surfaces

In addition to finding the net pressure force that a fluid exerts on an immersed surface, it is often important to know the total moment exerted by the fluid around some point. To find this, we remember the definition of a moment from mechanics,

$$\vec{M} = \vec{r}_\ell \times \vec{F}. \tag{2.24}$$

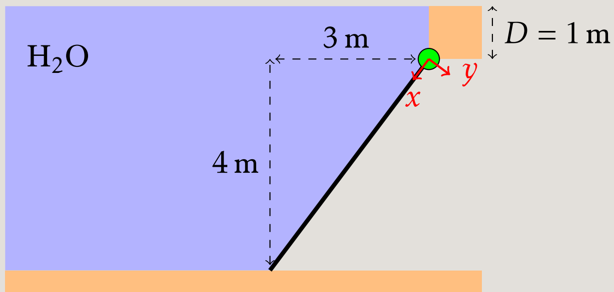
Here, $\vec{r}_\ell$ is the vector that begins at the point around which we are taking the moment and ends at the point of application of the force.

Since fluids exert forces continuously over surfaces, as described by Equation (2.4), we must consider the small differential moment generated by the differential pressure force exerted on a given differential section of a surface. That is,

$$\mathrm{d}\vec{M}_p = \vec{r}_\ell \times \mathrm{d}\vec{F}_p\,. \tag{2.25}$$

Again, r_ℓ is the vector from the point around which a moment is being taken to the differential surface on which the force is exerted—the point of application.

Example 2.2 : Force and moment on a planar gate



Consider the gate shown in the figure that is attached to a solid boundary with the green hinge. The gate is 2 m wide. What is the net pressure force on the gate and the net moment around the hinge that is caused by fluid pressure?

Solution:

We must start by defining a coordinate system. We will use Cartesian coordinates with the origin centred on the hinge. The x direction will be down the gate and the y direction will be normal to the gate and down to the right, as shown in the figure. This means the z axis will be out of the page.

The net pressure force will be the integral of all the differential pressure forces on the surface,

$$\mathrm{F}_p = \iint_S -p\,\hat{n}\,\mathrm{d}S\,.$$

We will be integrating in the x-z plane. We therefore need to find an expression for the pressure on the gate as a function of x and z. As water is incompressible, we can use $\Delta p = \rho g \Delta h$, where Δh is the fluid depth. A little trigonometry reveals that the depth as a function of x is $h = D + \frac{4}{5}x$. As the z direction is horizontal, the pressure is not a function of z. The pressure is therefore

$$p = \rho g\left(D + \frac{4}{5}x\right).$$

The unit normal that points into the fluid is $-\hat{j}$. The net pressure force is therefore

$$\vec{F}_p = \int_0^2 \int_0^5 \rho g\left(D + \frac{4}{5}x\right)\hat{j}\,\mathrm{d}x\,\mathrm{d}z\,.$$

The evaluation of this integral is straightforward.

To compute the moment around the hinge, we need to find $\vec{r}_\ell$, the distance from the hinge to the point of application. Because we put the origin on the hinge, that distance is simply the x-z coordinate,

$$\vec{r}_\ell = x\,\hat{i} + 0\,\hat{j} + z\,\hat{k}\,.$$

The differential moment generated on a differential surface of the gate is therefore

$$\begin{aligned} \mathrm{d}\vec{M}_p &= \vec{r}_\ell \times \mathrm{d}\vec{F}_p \\ &= \left(x\hat{\imath} + 0\hat{\jmath} + z\hat{k}\right) \times \rho g\left(D + \frac{4}{5}x\right)\hat{\jmath}\,\mathrm{d}S \\ &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ x & 0 & z \\ 0 & \rho g\left(D+\frac{4}{5}x\right)\mathrm{d}S & 0 \end{vmatrix} \\ &= -z\rho g\left(D + \frac{4}{5}x\right)\mathrm{d}S\,\hat{\imath} + x\rho g\left(D + \frac{4}{5}x\right)\mathrm{d}S\,\hat{k}. \end{aligned}$$

There are two components to the moment: one trying to rotate the gate around the x axis, and another trying to rotate the gate around the z axis. A perfect hinge does not allow any translation and only allows rotation around its primary axis. For us, this is the z axis. We will assume the hinge can safely resist any moment around the x axis and concentrate only on the z component of the generated moment,

$$\mathrm{d}M_z = x\rho g\left(D + \frac{4}{5}x\right)\mathrm{d}S.$$

We can integrate this function over the same limits as before to find the total moment around the z axis.

$$M_z = \int_0^2\int_0^5 x\rho g\left(D + \frac{4}{5}x\right)\mathrm{d}x\,\mathrm{d}z.$$

Again, this integral is simple to evaluate.

If the surface on which the fluid is exerting pressure forces is not planar, problems become more difficult. This is because the size of $\mathrm{d}S$ is not always clear and $\hat{n}$ varies over the surface. A technique is summarized here and shown in more detail in Section A.3 in Appendix A.

We evaluate these types of integrals by finding a parameterization of the curved surface through the definition of a vector function of two variables,

$$\vec{r}(u,v) = r_x(u,v)\,\hat{\imath} + r_y(u,v)\,\hat{k} + r_z(u,v)\,\hat{k}. \tag{2.26}$$

This function is defined such that, for all values between appropriate limits ($u_m in < u < u_m ax$ and $v_m in < v < v_m ax$), $\vec{r}$ gives a point on the surface and each point on the three dimensional surface corresponds to exactly one pair of u and v. The differential surface $\mathrm{d}S$ will have size

$$\mathrm{d}S = \left|\frac{\partial\vec{r}}{\partial u} \times \frac{\partial\vec{r}}{\partial v}\right|\mathrm{d}u\,\mathrm{d}v, \tag{2.27}$$

and the unit normal will be

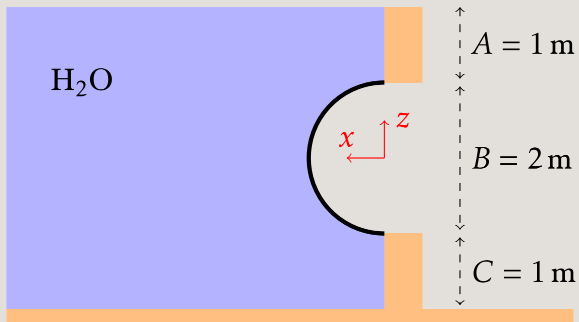
$$\hat{n} = \frac{\frac{\partial\vec{r}}{\partial u} \times \frac{\partial\vec{r}}{\partial v}}{\left|\frac{\partial\vec{r}}{\partial u} \times \frac{\partial\vec{r}}{\partial v}\right|}. \tag{2.28}$$

The product, $\hat{n}\,\mathrm{d}S$, which we use commonly, simplifies to

$$\hat{n}\,\mathrm{d}S = \left(\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right)\mathrm{d}u\,\mathrm{d}v\,. \tag{2.29}$$

Unfortunately, there is no easy way to ensure that the normal vector in Equation (2.29) points into the fluid. One must check each time to ensure the vector is in the correct direction. If it points out of the fluid, simply multiply by negative one.

Example 2.3 : Force on a hemispherical window



What is the net pressure force on the aquarium window shown above?

Solution:

The net pressure force is given by

$$\vec{F}_p = \iint_S -p\hat{n}\,\mathrm{d}S\,.$$

We will chose a mapping inspired by spherical coordinates,

$$x = \cos\theta\sin\phi \quad y = \sin\theta\sin\phi \quad z = \cos\phi\,,$$

or

$$\vec{r}(\theta,\phi) = \cos\theta\sin\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\phi\,\hat{k}\,,$$

with $0 < \phi < \pi$ and $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. The partial derivatives of this vector with respect to θ and ϕ are

$$\begin{aligned}
\frac{\partial \vec{r}}{\partial \theta} &= -\sin\theta\sin\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} + 0\,\hat{k} \\
\frac{\partial \vec{r}}{\partial \phi} &= \cos\theta\cos\phi\,\hat{i} + \sin\theta\cos\phi\,\hat{j} - \sin\phi\,\hat{k}\,.
\end{aligned}$$

We can therefore find $\hat{n}\,\mathrm{d}S$ as

$$\begin{aligned}
\hat{n}\,\mathrm{d}S &= \left(\frac{\partial\vec{r}}{\partial\theta}\times\frac{\partial\vec{r}}{\partial\phi}\right)\mathrm{d}\theta\,\mathrm{d}\phi \\
&= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \sin\theta\sin\phi & \cos\theta\sin\phi & 0 \\ \cos\theta\cos\phi & \sin\theta\cos\phi & -\sin\phi \end{vmatrix}\mathrm{d}\theta\,\mathrm{d}\phi \\
&= \left(-\cos\theta\sin^2\phi\,\hat{i} \;-\; \sin\theta\sin^2\phi\,\hat{j} \;-\; (\sin^2\theta\sin\phi\cos\phi+\cos^2\theta\sin\phi\cos\phi)\,\hat{k}\right)\mathrm{d}\theta\,\mathrm{d}\phi \\
&= \left(-\cos\theta\sin^2\phi\,\hat{i} \;-\; \sin\theta\sin^2\phi\,\hat{j} \;-\; \sin\phi\cos\phi\,\hat{k}\right)\mathrm{d}\theta\,\mathrm{d}\phi.
\end{aligned}$$

We must check if this vector points into or out of the fluid. We will do this by choosing one point on the surface and verifying the direction of $\hat{n}\,\mathrm{d}S$. The point we will use is $\theta = 0$ and $\phi = \frac{\pi}{2}$. This corresponds to the point $(1,0,0)$ in (x,y,z) space. The value of our $\hat{n}\,\mathrm{d}S$ at this point is

$$\hat{n}\,\mathrm{d}S = -\hat{i}.$$

This is in the wrong direction (out of the fluid). We must therefore multiply $\hat{n}\,\mathrm{d}S$ by negative one before proceeding,

$$\hat{n}\,\mathrm{d}S = \left(\cos\theta\sin^2\phi\,\hat{i} \;+\; \sin\theta\sin^2\phi\,\hat{j} \;+\; \sin\phi\cos\phi\,\hat{k}\right)\mathrm{d}\theta\,\mathrm{d}\phi.$$

The relative pressure in this situation is only a function of z and is given by

$$\begin{aligned}
p &= \rho g(A+\frac{B}{2}-z) \\
&= \rho g(A+\frac{B}{2}-\cos\phi).
\end{aligned}$$

We can now combine everything to determine the net pressure force on the window,

$$\begin{aligned}
\vec{F}_p &= \iint\limits_S -p\hat{n}\,\mathrm{d}S \\
&= \int_0^{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} -\rho g\left(A+\frac{B}{2}-\cos\phi\right)\left(\cos\theta\sin^2\phi\,\hat{i} \;+\; \sin\theta\sin^2\phi\,\hat{j} \;+\; \sin\phi\cos\phi\,\hat{k}\right)\mathrm{d}\theta\,\mathrm{d}\phi.
\end{aligned}$$

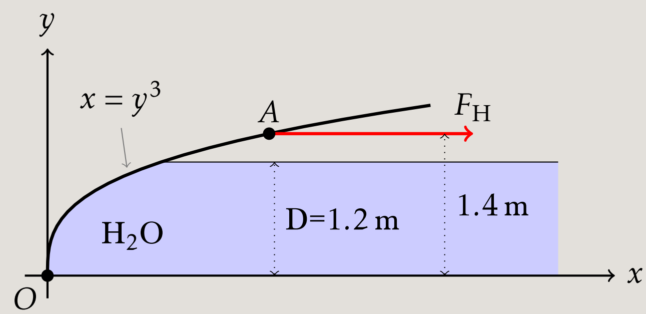
This vector expression can be split into its components, which are more easily integrated,

$$F_x = \int_0^{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} -\rho g\left(A+\frac{B}{2}-\cos\phi\right)\cos\theta\sin^2\phi\;\mathrm{d}\theta\,\mathrm{d}\phi = -2\pi\rho g,$$

$$F_y = \int_0^{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} -\rho g \left(A + \frac{B}{2} - \cos\phi \right) \sin\theta \sin^2\phi \, \mathrm{d}\theta \, \mathrm{d}\phi = 0 \,,$$

$$F_z = \int_0^{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} -\rho g \left(A + \frac{B}{2} - \cos\phi \right) \sin\phi \cos\phi \, \mathrm{d}\theta \, \mathrm{d}\phi = \frac{2}{3} \pi \rho g \,.$$

Example 2.4 : Moment on a curved gate



A gate of width, $W = 1.5\,\mathrm{m}$, is free to pivot about point O. What horizontal force, F_H, must be applied at point A to hold the gate in place?

Solution:

The easiest parameterization for this question is

$$\vec{r} = u^3 \hat{i} + u \hat{j} + v \hat{k}$$

with $0 < u < 1.2\,\mathrm{m}$ and $0 < v < 1.5\,\mathrm{m}$. The derivatives of this vector with respect to u and v are

$$\frac{\partial \vec{r}}{\partial u} = 3u^2 \hat{i} + \hat{j} \,,$$
$$\frac{\partial \vec{r}}{\partial v} = \hat{k} \,.$$

We can use this to find $\hat{n}\mathrm{d}S$ as

$$\begin{aligned} \hat{n}\mathrm{d}S &= \left(\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right) \mathrm{d}u \, \mathrm{d}v \\ &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3u^2 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \mathrm{d}u \, \mathrm{d}v \\ &= (\hat{i} - 3u^2 \hat{j}) \mathrm{d}u \, \mathrm{d}v \,. \end{aligned}$$

This vector points into the fluid, so it is in the direction we want.

The pressure in this situation is given by

$$\begin{aligned} p &= \rho g(D-y) \\ &= \rho g(D-u). \end{aligned}$$

This can be used to find the differential pressure on each differential surface,

$$\begin{aligned} \mathrm{d}\vec{F}_p &= -p\,\hat{n}\,\mathrm{d}S \\ &= -\left[\rho g(D-u)\right](\hat{\imath}-3u^2\hat{\jmath})\mathrm{d}u\,\mathrm{d}v \end{aligned}$$

The moment generated by each differential force follows the relation

$$\mathrm{d}\vec{M}_p = \vec{r}_\ell \times \mathrm{d}\vec{F}_p\,.$$

Since the hinge is on the origin, $\vec{r}_\ell$ (the vector from the origin to the point of application of the force) is just the coordinate,

$$\vec{r}_\ell = \vec{r} = u^3\hat{\imath} + u\hat{\jmath} + v\hat{k}\,.$$

Therefore,

$$\begin{aligned} \mathrm{d}\vec{M}_p &= \vec{r}_\ell \times \mathrm{d}\vec{F}_p \\ &= -\rho g(D-u)\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ u^3 & u & v \\ 1 & -3u^2 & 0 \end{vmatrix}\mathrm{d}u\,\mathrm{d}v \\ &= -\rho g(D-u)\left[3u^2v\hat{\imath} + v\hat{\jmath} - (3u^5+u)\hat{k}\right]\mathrm{d}u\,\mathrm{d}v\,. \end{aligned}$$

There are three components to the moment generated by the pressure acting on a differential surface. However, the hinge in the problem is only free to rotate about the z axis. We are therefore only concerned with the $\hat{k}$ component of $\mathrm{d}\vec{M}_p$. The total moment about this axis is

$$\begin{aligned} M_z &= \int_0^W\int_0^D \rho g(D-u)(u+3u^5)\,\mathrm{d}u\,\mathrm{d}v \\ &= \rho g W\left(\frac{D^3}{6}+\frac{D^7}{14}\right) \\ &= 8004\,\mathrm{N\,m}. \end{aligned}$$

To resist this moment, the applied force must generate an equally large moment in the opposite direction,

$$M_z - F(1.4\,\mathrm{m}) = 0\,.$$

This gives

$$F = \frac{M_z}{1.4\,\mathrm{m}} = 5.7\,\mathrm{kN}\,.$$

2.5 Buoyancy

The force exerted by a fluid on a solid object that is immersed within it is called the *buoyant* force. Let's investigate the nature of this force. Imagine an object immersed in a fluid, as shown in Figure 2.1. The object occupies a volume, V, and is enclosed in the closed surface, S. The fluid exerts a net force on the object that is due to the fluid pressure. This force is given by

$$\vec{F}_p = \oiint_S -p\,\hat{n}\,\mathrm{d}S\,. \tag{2.30}$$

As the surface encloses a solid shape, it must be a closed surface. We can obviously move the negative sign out of the integral,

$$\vec{F}_p = -\oiint_S p\,\hat{n}\,\mathrm{d}S\,. \tag{2.31}$$

Now, I would like to use the gradient theorem (Equation (A.30) of Section A.7 from Appendix A) to transform this surface integral to a volume integral. Initially, this seems fine, as Equation (2.31) gives an integral that is of exactly the same form as the surface integral in Equation (A.30). However, the gradient theorem requires that the scalar field be defined throughout the enclosed volume. For us, this is not the case. The relation for the pressure field is only valid in the fluid. It cannot be said to give us the "fluid pressure" within a solid body.

What we can do is imagine removing the solid body and replacing it with the fluid that would have been there. At this point, the integral on the right-hand side of Equation (2.31) can no longer be interpreted as giving a force (as there is no solid surface on which a pressure force can be exerted). However, it remains a valid mathematical operation, the result of which is the pressure force that would be exerted if the body were returned. The pressure field is now defined within the volume and the gradient theorem can be used. This gives

$$\vec{F}_p = -\oiint_S p\,\hat{n}\,\mathrm{d}S = -\iint_V \vec{\nabla} p\,\mathrm{d}V\,. \tag{2.32}$$

We can now use Equation (2.12) to replace the gradient of the pressure field with $\rho\vec{g}$,

$$\vec{F}_p = -\iiint_V \rho\vec{g}\,\mathrm{d}V\,. \tag{2.33}$$

In this relation, ρ is the density of the fluid that *would have been there if the object were not.* The integral gives the weight of the volume of fluid displaced by the object. The negative sign reversed the direction of the resulting force. This is known as Archimedes' principle.

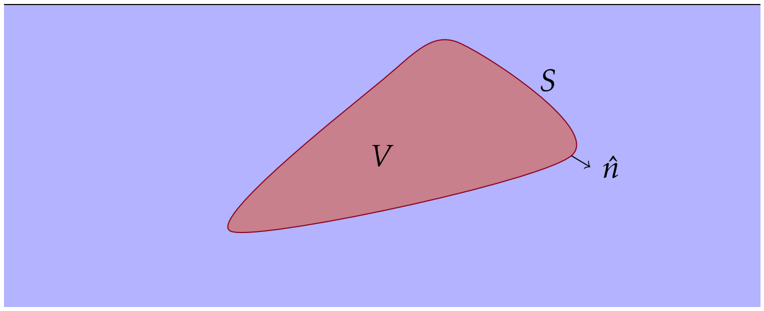


Figure 2.1: Solid object immersed in a fluid, subject to a buoyant force.

Archimedes' principle

The buoyant force exerted on an object immersed in a fluid is equal in magnitude to the weight of the displaced fluid, but acts in the opposite direction.

Example 2.5 : Buoyancy example

2.6 Fluids in rigid-body motion

By *fluid in rigid-body motion*, we mean a fluid that appears to be at rest in a particular frame of reference. This frame of reference, however, does not need to be an inertial frame of reference. It can be undergoing an acceleration.

The derivation follows a similar path to that demonstrated in Section 2.3. We image that we are in the frame of reference of an arbitrary closed volume of fluid. Now, rather than the sum of the forces acting on the fluid giving zero, it will give the mass times the acceleration (Newton's second law),

$$\sum \vec{F} = \iiint_V \rho \vec{a} \, \mathrm{d}V \, . \tag{2.34}$$

Here $\vec{a}$ is the acceleration field of the frame of reference with respect to an inertial frame. We must use the integral on the right-hand side because the acceleration of each point in our frame of reference will not be the same if it is in rotation. There are only two forces acting on the fluid, weight and the force due to fluid pressure. This means

$$\iiint_V \rho \vec{g} \, \mathrm{d}V + \oiint_S -p \, \hat{n}_{\text{out}} \, \mathrm{d}S = \iiint_V \rho \vec{a} \, \mathrm{d}V \, . \tag{2.35}$$

In the second term, corresponding to the pressure force, we are using the unit normal that points out of the volume rather than into the fluid. This is because we are interested in the force that the wall of the container exerts on the fluid, rather than the force that the fluid exerts on the wall. We now use the gradient theorem, shown in Section A.7 of Appendix A, to transform the surface integral into a volume integral,

$$\iiint_V \rho\vec{g}\,\mathrm{d}V + \iiint_V -\vec{\nabla}p\,\mathrm{d}V = \iiint_V \rho\vec{a}\,\mathrm{d}V\,. \tag{2.36}$$

We can now combine all the integrals, as they are over the same volume,

$$\iiint_V \left[\rho\vec{g} - \vec{\nabla}p - \rho\vec{a}\right]\mathrm{d}V = 0\,. \tag{2.37}$$

Since the volume is arbitrary, the integral must give zero over any volume. The only way this can happen is if the integrand is zero everywhere,

$$\rho\vec{g} - \vec{\nabla}p - \rho\vec{a} = 0\,. \tag{2.38}$$

This can be simplified slightly to get the final form of the expression.

Variation of fluid pressure in an accelerating frame

If a fluid is “at rest” in an accelerating frame of reference, the pressure field changes according to the relation

$$\vec{\nabla}p = \rho(\vec{g} - \vec{a})\,. \tag{2.39}$$

Equation (2.39) is a generalization of Equation (2.12) for fluids that are “at rest” in accelerating frames of reference. It is clear to see that, if $\vec{a}$ is zero, Equation (2.12) is recovered.

2.7 Review

In this chapter we learned two fundamental relations. We learned that pressure of a fluid in contact with a solid boundary will generate a force on the boundary following the relation

$$\mathrm{d}\vec{F}_p = -p\hat{n}\,\mathrm{d}S\,.$$

We will see that this relation for the pressure force exerted by a fluid is actually true even for fluid that are in motion.

The second relation we learned was how the fluid pressure varies in space when the fluid it “at rest” in a frame of reference,

$$\vec{\nabla}p = \rho(\vec{g} - \vec{a})\,.$$

This relation is only true for fluid that are undergoing rigid-body motions. It is not generally correct for fluids that are flowing.

3 Fluid Dynamics: The Integral Approach

There are two main ways fluids can be modelled mathematically. The first is the integral approach, where we define control volumes and track how properties in the control volume change due to fluid fluxes and external influences. The second is a differential treatment, where we define partial differential equations (PDEs), that describe the evolution of a fluid flow. Both methods express the same concepts in different forms. The treatment that is most appropriate will vary depending on the situation and what information about the flow is being sought. In this chapter, we are introduced to the integral approach to fluid dynamics.

3.1 The velocity field

A velocity field of a fluid is a vector field that, at every point in space and time, gives the velocity of the fluid at that point and instant.

Velocity Field

A vector function that takes position and time as inputs and returns the velocity at that point and instant, $\vec{v}(\vec{x}, t)$.

3.1.1 Streamlines and streamtubes

A streamline is a line that is always tangent to a velocity field. This means that at every point along the line, the velocity is parallel to the line. An example of several streamlines within a velocity field is shown in Figure 3.1. Once can see that the streamlines, shown in blue, are always parallel to the local fluid velocity.

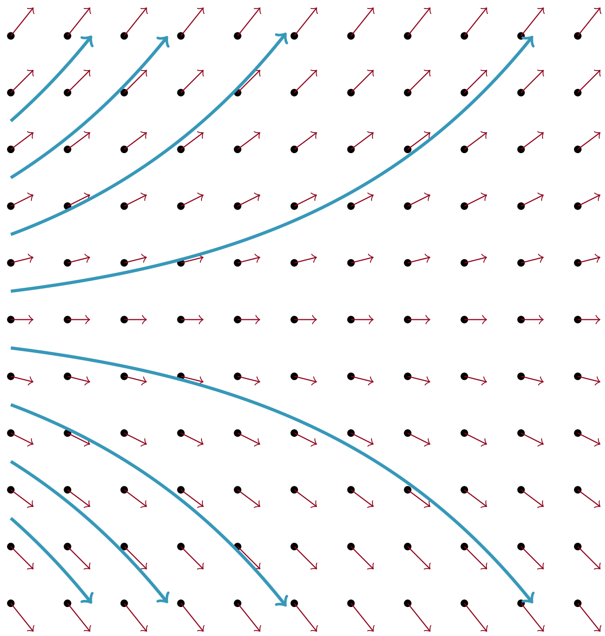


Figure 3.1: Streamlines within a two-dimensional vector field.

Streamline

A streamline is a line that is parallel to the velocity field at every point.

It should be noted that, in general, individual elements of the fluid do not follow streamlines as they move. This is because the streamlines are a represenation of the velocity field at a particular point in time. For time-varying flows, the streamlines change from instant to instant. Only in the case of steady-state, when the streamlines are constant, will the fluid follow the paths that they describe.

a streamtube is a tube whose walls are composed entirely of streamlines. As the fluid velocity must be parallel to streamlines, the fluid's velocity must always be tangent to the sides of a streamtube. This means the normal component of the fluid velocity at the wall of a streamtube is zero. Thus, there is never any flux of fluid through the sides of a streamtube. Any fluid that enters a streamtube at one end, must eventually exit at the other end.

Streamtube

A streamtube is a tube whose walls are composed entirely of streamlines. As a fluid cannot cross a streamline, fluid will never exit a streamtube through its sides.

3.2 Lagrangian vs. Eulerian descriptions

The laws of mechanics with which you are familiar up to now apply directly to a fixed mass. For example, Newton's second law can be written as

$$\begin{aligned}\sum \vec{F} &= m\vec{a} \\ &= m\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}(m\vec{v}) \\ &= \frac{\mathrm{d}\vec{P}}{\mathrm{d}t}.\end{aligned} \tag{3.1}$$

Here, $\vec{P}$, is the momentum. We can see that the sum of the forces is equal to the mass times the acceleration, which can also be expressed as the rate of change of momentum.

If we apply Equation (3.1) to a fixed mass, we can find where the mass will be at any time in the future. The position is a dependant variable. This is what is known as a *Lagrangian description* of the system.

These Lagrangian treatments are usually not the most convenient way to describe fluid flows. This is because, by their nature, fluids are usually flowing. A fixed mass of fluid will be continuously deforming and moving through any system. It is very inconvenient to try to keep track of the position and state of each mass of fluid. Usually, when working with fluids, we are only interested in properties at fixed locations in space. For example, what is the distribution of fluid pressure on a surface? It doesn't matter that, at every instant, the actual mass of fluid that is in contact with the surface is different. Such descriptions, in which position is an independent variable, is called an *Eulerian treatment.*

Lagrangian and Eulerian descriptions

The laws on mechanics can be expressed in several forms. The two most common forms are

Lagrangian description: Laws apply directly to a fixed mass. Position is a dependant variable.

Eulerian description: Laws apply directly to a fixed point or volume in space. Position is an independent variable.

3.2.1 A Lagrangian expression of the conservation laws

As we saw in Section 3.2, a Lagrangian treatment follows a fixed mass. For this course, we will use three laws for conserved quantities:

1. Conservation of mass,

2. Conservation of momentum,

3. Conservation of energy.

We will express these laws by defining a volume, V_m, to be the volume that encloses a chosen mass of fluid and follows it through space. The volume, therefore, changes with time. The mass contained in this volume is

$$\iiint\limits_{V_m(t)} \rho \, \mathrm{d}V\,.$$

The conservation of mass says that

$$\frac{\mathrm{d}}{\mathrm{d}t} \iiint\limits_{V_m(t)} \rho \, \mathrm{d}V = 0\,. \tag{3.2}$$

The conservation of momentum is

$$\begin{aligned} \sum \vec{F} &= \frac{\mathrm{d}\vec{P}}{\mathrm{d}t} \\ &= \frac{\mathrm{d}}{\mathrm{d}t} \iiint\limits_{V_m(t)} \rho \vec{v} \, \mathrm{d}V\,. \end{aligned} \tag{3.3}$$

Here, $\vec{P}$ is the momentum. The rate of change of momentum with time is equal to the sum of the applied forces.

The law of conservation of energy is

$$\begin{aligned} \dot{Q} - \dot{W} &= \frac{\mathrm{d}E}{\mathrm{d}t} \\ &= \frac{\mathrm{d}}{\mathrm{d}t} \iiint\limits_{V_m(t)} \rho e \, \mathrm{d}V\,. \end{aligned} \tag{3.4}$$

Here, $\dot{Q}$ is the rate of heat transfer to the mass, $\dot{W}$ is the rate a which work is done on the environment*, E is the energy, and e is the specific energy.

We can notice that each of these conserved quantities (mass, momentum, and energy) is the integral of some field, η, times the mass density,

$$N = \iiint\limits_{V_m(t)} \rho \eta \, \mathrm{d}V\,. \tag{3.5}$$

This is shown in Table 3.1.

*The negative sign on the $\dot{W}$ term of Equation (3.4) comes from the fact that, by tradition, this term is defined as the work done on the surroundings. Therefore, when $\dot{W}$ is positive, energy is lost.

Table 3.1: Correspondence between conserved quantity and integrated field,

$$N = \iiint\limits_{V_m(t)} \rho\eta \, \mathrm{d}V\,.$$

Conserved quantity, N	Corresponding field, η
Mass, M	1
Momentum, $\vec{P}$	$\vec{v}$
Energy, E	e

3.3 Reynolds transport theorem

The “recipe” that we will use to recast our Lagrangian laws of mechanics in an Eulerian setting is known as Reynolds transport theorem [3]. This theorem shows how to differentiate the integral of a field when the volume of integration is changing with time. As a review, we remember that, for a function, f, that varies in space and time, if the volume of integration is constant in time, one can write

$$\frac{\mathrm{d}}{\mathrm{d}t} \iiint\limits_{V} f(x,y,z,t) \; \mathrm{d}V = \iiint\limits_{V} \frac{\partial}{\partial t} f(x,y,z,t) \; \mathrm{d}V\,. \tag{3.6}$$

This is possible because the volume of integration is independent of time. Thus, the operations are independent and the order can be exchanged. However, if the volume of integration changes with time, this is not the case. Reynolds transport theorem allows us to treat such cases,

$$\frac{\mathrm{d}}{\mathrm{d}t} \iiint\limits_{V(t)} f(x,y,z,t) \; \mathrm{d}V = ?$$

To do this, we return to the definition of a derivative,

$$\frac{\mathrm{d}}{\mathrm{d}t} g(t) = \lim_{\Delta t \to 0} \frac{g(t+\Delta t) - g(t)}{\Delta t}\,. \tag{3.7}$$

For us, the function, g, is the integral of f over the time-varying volume,

$$\frac{\mathrm{d}}{\mathrm{d}t} \iiint\limits_{V(t)} f(x,y,z,t) \; \mathrm{d}V = \lim_{\Delta t \to 0} \left[\frac{\iiint\limits_{V(t+\Delta t)} f(x,y,z,t+\Delta t) \; \mathrm{d}V - \iiint\limits_{V(t)} f(x,y,z,t) \; \mathrm{d}V}{\Delta t} \right]. \tag{3.8}$$

This is illustrated in Figure 3.2. We will evaluate the integral that is evaluated at $t+\Delta t$ in several steps,

1. Evaluate the integral of $f(x,y,z,t+\Delta t)$ over the space occupied by the volume at time, t, illustrated in orange and pink.

2. Add the integral of $f(x,y,z,t+\Delta t)$ over the area covered by the volume at $t+\Delta t$ but not at t. These regions are illustrated in blue.

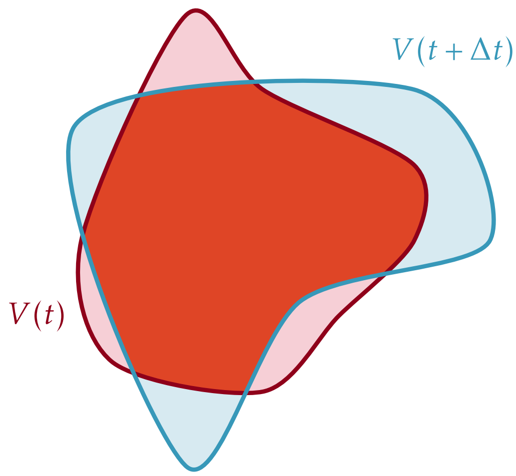


Figure 3.2: Graphical representation of Reynolds transport theorem. The orange area is occupied by the volume at times t and $t+\Delta t$. The blue areas are occupied by the volume at $t+\Delta t$, but were not covert at t. The pink areas were covered by the volume at t but are no longer covered at $t+\Delta t$.

3. Subtract the integral of $f(x,y,z,t+\Delta t)$ over the area that was covered by the volume at t, but that is not covered at $t+\Delta t$. These regions are illustrated in pink.

For step one, the expression of the integral of $f(x,y,z,t+\Delta t)$ over the space occupied by the volume at time, t, is straight forward,

$$\iiint_{V(t)} f(x,y,z,t+\Delta t)\ \mathrm{d}V\,.$$

It turns out, we can add and subtract the integral over the regions from steps two and three in one operation. To do this, we define a new variable, τ. Like we did in Chapter 2, we assume that at, any fixed time, our surface can be parameterized in terms of two variables u and v. However, this parameterization will also be a function of τ such that, for $\tau = 0$, the parameterization describes the surface at time, t, and at $\tau = \Delta t$ the parameterization describes the surface at $t+\Delta t$.

The volume swept out by a infinitesimal part of the surface, $\mathrm{d}S$, as it moves from some value, τ, to $\tau + \mathrm{d}\tau$ is

$$(\vec{v}_b\,\mathrm{d}\tau)\cdot\hat{n}\,\mathrm{d}S = \vec{v}_b\cdot\hat{n}\,\mathrm{d}S\,\mathrm{d}\tau\,. \tag{3.9}$$

Here, $\vec{v}_b$ is the velocity of the boundary and $\hat{n}$ is the outward-facing unit normal. The distance that the surface element moves over $\mathrm{d}\tau$ is $(\vec{v}_b\,\mathrm{d}\tau)$. It is only the component of this displacement that is normal to the surface which actually sweeps out volume—this is why we use the dot product.

The size the differential surface element, $\mathrm{d}S$, its velocity, $\vec{v}_b$, and the unit normal, $\hat{n}$, are all functions of τ. This is because they all vary as the surface moves from its original position at $\tau = 0$ to its final position at $\tau = \Delta t$.

Because we define $\hat{n}$ to be outward-facing, the dot product in Equation 3.9 will be positive in the blue regions of Figure 3.2 and negative in the pink regions. The difference in the magnitude of the volume at time, $t+\Delta t$, as compared to at time, t, is therefore

$$|V(t+\Delta t)| - |V(t)| = \int_0^{\Delta t} \oiint_S \vec{v}_b \cdot \hat{n}\, \mathrm{d}S\, \mathrm{d}\tau \tag{3.10}$$

To accomplish steps two and three, from the list above, we integrate the function, evaluated at $t+\Delta t$, over this volume,

$$\int_0^{\Delta t} \oiint_S f(x,y,z,t+\Delta t)\, \vec{v}_b \cdot \hat{n}\, \mathrm{d}S\, \mathrm{d}\tau\,. \tag{3.11}$$

It should be kept in mind that, even though we are integrating over the region swept out by the surface as it moves from its position at t to $t+\Delta t$, the function is always evaluated at $t+\Delta t$—as needing in Equation (3.8).

We can now express the integral of $f(x,y,z,t+\Delta t)$ over the position of the volume at $t+\Delta t$ as

$$\begin{aligned}\iiint\limits_{V(t+\Delta t)} f(x,y,z,t+\Delta t)\, \mathrm{d}V = &\iiint\limits_{V(t)} f(x,y,z,t+\Delta t)\ \mathrm{d}V \\ &+ \int_0^{\Delta t} \oiint_S f(x,y,z,t+\Delta t)\, \vec{v}_b \cdot \hat{n}\, \mathrm{d}S\, \mathrm{d}\tau\,.\end{aligned} \tag{3.12}$$

Now, we use the mean-value theorem for definite integrals (shown in Section A.2.1 of Appendix A)) to rewrite the final integral,

$$\begin{aligned}\iiint\limits_{V(t+\Delta t)} f(x,y,z,t+\Delta t)\, \mathrm{d}V = &\iiint\limits_{V(t)} f(x,y,z,t+\Delta t)\ \mathrm{d}V \\ &+ \Delta t \left[\oiint_S f(x,y,z,t+\Delta t)\, \vec{v}_b \cdot \hat{n}\, \mathrm{d}S\right]^{\star}\,.\end{aligned} \tag{3.13}$$

The star denotes the evaluation of the integral at the value of $\tau = \tau^{\star}$, for which it gives its average value over the interval $0 < \tau < \Delta t$.

We can substitute Equation (3.13) into Equation (3.8) to get

$$\frac{\mathrm{d}}{\mathrm{d}t} \iiint\limits_{V(t)} f(x,y,z,t)\ \mathrm{d}V =$$

$$\lim_{\Delta t \to 0} \frac{\iiint\limits_{V(t)} f(x,y,z,t+\Delta t)\ \mathrm{d}V + \Delta t \left[\oiint_S f(x,y,z,t+\Delta t)\, \vec{v}_b \cdot \hat{n}\, \mathrm{d}S\right]^{\star} - \iiint\limits_{V(t)} f(x,y,z,t)\ \mathrm{d}V}{\Delta t}\,.$$

This can be simplified by combining the first and last integral as

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint\limits_{V(t)} f(x,y,z,t)\ \mathrm{d}V =$$

$$\lim_{\Delta t\to 0}\left\{\iiint\limits_{V(t)} \frac{f(x,y,z,t+\Delta t)-f(x,y,z,t)}{\Delta t}\ \mathrm{d}V + \left[\oiint\limits_{S} f(x,y,z,t+\Delta t)\,\vec{v}_b\cdot\hat{n}\,\mathrm{d}S\right]^{\star}\right\}.$$

In the limit, the first term is obviously just the integral of the derivative of f. As we take the limit of Δt going to zero, the second term simply becomes the surface integral evaluated at t. This gives the final form of Reynolds transport theorem.

Reynolds transport theorem

Reynolds transport theorem allows us to move the derivative of the integral of a function over a time-varying domain inside of the integral sign,

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint\limits_{V(t)} f\ \mathrm{d}V = \iiint\limits_{V(t)} \frac{\partial f}{\partial t}\,\mathrm{d}V + \oiint\limits_{S(t)} f\ \vec{v}_b\cdot\hat{n}\,\mathrm{d}S\,. \tag{3.14}$$

Here, $\vec{v}_b$ is the velocity of the boundary, S, enclosing the volume. This velocity of the surface can be different at every point. The vector, $\hat{n}$, is the outward-facing unit normal. The function, f, can vary in all space dimensions and time.

3.3.1 A useful form of Reynolds transport theorem

As we said, our goal in using the Reynolds transport theorem is to transform our Lagrangian laws of mechanics to an Eulerian treatment. To do this, we first realize that the conservation laws expressed in Section 3.2.1 can be written using Equation (3.14) and choosing the volume to be that which encloses (and moves with) a fixed mass of fluid, V_m. The velocity of the boundary of this volume at every point is simply the velocity of the fluid, $\vec{v}_b = \vec{v}$. The function, f, is chosen to be one of the η from Table 3.1,

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint\limits_{V_m} \rho\eta\ \mathrm{d}V = \iiint\limits_{V_m} \frac{\partial}{\partial t}(\rho\eta)\,\mathrm{d}V + \oiint\limits_{S_m} \rho\eta\ \vec{v}\cdot\hat{n}\,\mathrm{d}S\,. \tag{3.15}$$

We then chose a second volume, V. This second volume is chosen so that it occupies exactly the same space as V_m at an arbitrary instant, but can be undergoing any motion and deformation,

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint\limits_{V} \rho\eta\ \mathrm{d}V = \iiint\limits_{V} \frac{\partial}{\partial t}(\rho\eta)\,\mathrm{d}V + \oiint\limits_{S} \rho\eta\ \vec{v}_b\cdot\hat{n}\,\mathrm{d}S\,. \tag{3.16}$$

The velocity of the boundary of this second volume is $\vec{v}_b$.

Subtracting Equation (3.16) from Equation (3.15) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint\limits_{V_m} \rho\eta\ \mathrm{d}V - \frac{\mathrm{d}}{\mathrm{d}t}\iiint\limits_{V} \rho\eta\ \mathrm{d}V =$$
$$\iiint\limits_{V_m} \frac{\partial}{\partial t}(\rho\eta)\,\mathrm{d}V - \iiint\limits_{V} \frac{\partial}{\partial t}(\rho\eta)\,\mathrm{d}V + \oiint\limits_{S_m} \rho\eta\,\vec{v}\cdot\hat{n}\,\mathrm{d}S - \oiint\limits_{S} \rho\eta\,\vec{v}_b\cdot\hat{n}\,\mathrm{d}S\,. \tag{3.17}$$

Since, at the instant considered, the two volumes cover the same space, the integrals on the right-hand side can be combined. This is not the case on the left hand side, because the rate of change of the two volumes is not equal and these integrals are differentiated in time. The result is

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint\limits_{V_m} \rho\eta\ \mathrm{d}V - \frac{\mathrm{d}}{\mathrm{d}t}\iiint\limits_{V} \rho\eta\ \mathrm{d}V = \iiint\limits_{V}\left(\frac{\partial}{\partial t}(\rho\eta) - \frac{\partial}{\partial t}(\rho\eta)\right)\mathrm{d}V + \oiint\limits_{S} \rho\eta\,(\vec{v}-\vec{v}_b)\cdot\hat{n}\,\mathrm{d}S\,. \tag{3.18}$$

The first integral on the right hand side is zero,

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint\limits_{V_m} \rho\eta\ \mathrm{d}V - \frac{\mathrm{d}}{\mathrm{d}t}\iiint\limits_{V} \rho\eta\ \mathrm{d}V = \oiint\limits_{S} \eta\,(\vec{v}-\vec{v}_b)\cdot\hat{n}\,\mathrm{d}S\,. \tag{3.19}$$

In the situation when the second volume is constant, as it must be for an Eulerian description, $\vec{v}_b$ is zero. This gives a useful version of Reynolds transport theorem that we will use throughout this course.

A useful form of Reynolds transport theorem

Throughout this course, we will use a particular form of Reynolds transport theorem,

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint\limits_{V_m} \rho\eta\ \mathrm{d}V = \frac{\mathrm{d}}{\mathrm{d}t}\iiint\limits_{V} \rho\eta\ \mathrm{d}V + \oiint\limits_{S} \rho\eta\,\vec{v}\cdot\hat{n}\,\mathrm{d}S\,. \tag{3.20}$$

Here, V_m is the volume that moves with a fixed mass, V is a stationary control volume, S is the boundary of the stationary volume, $\vec{v}$ is the fluid velocity. One chooses η appropriately depending on which conservation law is being used, as shown in Table 3.1.

The three terms in Equation (3.20) have very clear physical meanings. The term on the left is the rate of change of some property (*e.g.* mass, momentum, or energy) of a fixed mass. This is divided on the right-hand side between how much of that change is added to the fixed, Eulerian control volume, and how much is carried out of the control volume by the fluid flow.

3.4 Conservation of mass

The first law that we will translate from our traditional Lagrangian treatment to an Eulerian treatment is the law of *conservation of mass.* This law is found by chosing $\eta = 1$ in Equation (3.20). The left hand side of the equation is equal to zero, as shown in Equation (3.2). Therefore, the Eulerian form of the mass conservation equation is

$$0 = \frac{\mathrm{d}}{\mathrm{d}t} \iiint_V \rho \ \mathrm{d}V + \oiint_S \rho \, \vec{v} \cdot \hat{n} \, \mathrm{d}S \, . \tag{3.21}$$

This equation says that the time rate of change of mass within a chosen control volume (the first term) plus the flux of mass out of the control volume (the second term) must sum to zero. Physically, this makes sense. The only way the amount of mass can change within a volume, is if it crossed the surface and either enters or leaves.

3.5 Conservation of momentum

The Eulerian form of the momentum equation is found by chosing η to be the fluid velocity, $\vec{v}$, in Equation (3.20). By Equation (3.3), the time rate of change of momentum for a fixed mass (the left hand side) is equal to the sum of the applied forces. This gives the relation

$$\sum \vec{F} = \frac{\mathrm{d}}{\mathrm{d}t} \iiint_V \rho \vec{v} \ \mathrm{d}V + \oiint_S \rho \vec{v} \, \vec{v} \cdot \hat{n} \, \mathrm{d}S \, . \tag{3.22}$$

One way to interpret Equation (3.22) is to consider forces as a source of momentum. All the forces exerted on the control volume will generate a given amount of momentum within the control volume. This will either cause the amount of momentum within the control voluem to change with time (the first term on the right-hand side), or will be carried out of the control volume by the flow (the final term).

3.6 Conservation of momentum in moving frames

The Lagrangian form of Newton's second law expressed in an accelerating frame of reference can be written as

$$\sum \vec{F} = \frac{\mathrm{d}}{\mathrm{d}t} \iiint_{V_m(t)} \rho \vec{v} \mathrm{d}V + \iiint_{V_m(t)} \rho \vec{a}_{\mathrm{ref}} \mathrm{d}V \, . \tag{3.23}$$

In this relation, $\vec{a}_{\mathrm{ref}}$ is the acceleration of the frame of reference. We can rearrange this expression to give

$$\sum \vec{F} - \iiint_{V_m(t)} \rho \vec{a}_{\mathrm{ref}} \mathrm{d}V = \frac{\mathrm{d}}{\mathrm{d}t} \iiint_{V_m(t)} \rho \vec{v} \mathrm{d}V \, . \tag{3.24}$$

Again, we set $\eta = \vec{v}$ in Equation (3.20) and we can obtain the Eulerian form of the conservation of momentum in an accelerating frame of reference,

$$\sum \vec{F} - \iiint\limits_{V_m(t)} \rho \vec{a}_{\mathrm{ref}} \, \mathrm{d}V = \frac{\mathrm{d}}{\mathrm{d}t} \iiint\limits_{V} \rho \vec{v} \, \mathrm{d}V + \oiint\limits_{S} \rho \vec{v} \, \vec{v} \cdot \hat{n} \, \mathrm{d}S \,. \tag{3.25}$$

Again, at any point when we want to apply this equation, we chose the volume, V_m, to be the volume that contains the mass within the control volume at that time. We can therefore swap the integral of the second term to be over the control volume,

$$\sum \vec{F} - \iiint\limits_{V} \rho \vec{a}_{\mathrm{ref}} \, \mathrm{d}V = \frac{\mathrm{d}}{\mathrm{d}t} \iiint\limits_{V} \rho \vec{v} \, \mathrm{d}V + \oiint\limits_{S} \rho \vec{v} \, \vec{v} \cdot \hat{n} \, \mathrm{d}S \,. \tag{3.26}$$

It should be noted that the integrals over the control volume in this expression must take into account *all mass* within the control volume, not only the fluid.

3.7 The Bernoulli equation

The final relation that will be important in our control-volume analysis of fluid dynamics is the Bernoulli equation. To derive this equation, we will apply Equation (3.22) to an infinitessimal control volume that is a smooth strem tube. We chose a volume that is small enough that the direction of fluid velocity in our stream tube can be considered to be straight. We will restric out analysis to the situation where there are no viscous forces, the flow is in steady-state, and there are no external sources of energy.

We first apply the conservation of mass relation to this control volume. As we are in steady state, this gives

$$\begin{aligned} 0 &= \oiint\limits_{S} \rho \vec{v} \cdot \hat{n} \, \mathrm{d}S \\ &= \iiint\limits_{V} \vec{\nabla} \cdot (\rho \vec{v}) \, \mathrm{d}V \,. \end{aligned}$$

Here, the divergence theorem has been used. As the integral must be zero for any arbitrary infinitessimal stream tube, the integrand must always be zero,

$$\vec{\nabla} \cdot (\rho \vec{v}) = 0. \tag{3.27}$$

We now apply the law of conservation of momentum to the control volume. There are two forces acting on the fluid within our tube: weight and pressure forces. The weight is simply

$$\vec{F}_g = \iiint\limits_{V} \rho \vec{g} \, \mathrm{d}V.$$

The pressure force acting on our volume will be the integral of $-p\hat{n}\,\mathrm{d}S$ over the entire surface of the volume,

$$\begin{aligned}\vec{F}_p &= \oiint_S -p\hat{n}\;\mathrm{d}S \\ &= -\iiint_V \vec{\nabla}p\,\mathrm{d}V.\end{aligned}$$

Here, the gradient theorem has been used. As we are in steady state, Equation (3.22) is therefore

$$\iiint_V \rho\vec{g}\,\mathrm{d}V - \iiint_V \vec{\nabla}p\,\mathrm{d}V = \oiint_S \rho\vec{v}\,\vec{v}\cdot\hat{n}\,\mathrm{d}S\,. \tag{3.28}$$

We now define the unit vector $\hat{s}$ to be the vector of unit length that points in the direction of the fluid flow. By taking the dot product of Equation (3.28) with $\hat{s}$, we can look at the component in the direction of fluid flow,

$$\begin{gathered}\hat{s}\cdot\iiint_V \rho\vec{g}\,\mathrm{d}V - \hat{s}\cdot\iiint_V \vec{\nabla}p\,\mathrm{d}V = \hat{s}\cdot\oiint_S \rho\vec{v}\,\vec{v}\cdot\hat{n}\,\mathrm{d}S \\ \iiint_V \rho(\hat{s}\cdot\vec{g})\,\mathrm{d}V - \iiint_V (\hat{s}\cdot\vec{\nabla}p)\,\mathrm{d}V = \oiint_S \rho|v|\,\vec{v}\cdot\hat{n}\,\mathrm{d}S\,.\end{gathered}$$

Because $\vec{v}$ and $\hat{s}$ point int the same direction, their dot product simply gives the magnitude of the fluid velocity.

We can use the divergence theorem on the last term, which yields

$$\iiint_V \rho(\hat{s}\cdot\vec{g})\,\mathrm{d}V - \iiint_V (\hat{s}\cdot\vec{\nabla}p)\,\mathrm{d}V = \iiint_V \vec{\nabla}\cdot(\rho|v|\,\vec{v})\,\mathrm{d}V\,.$$

This can be combined into one integral,

$$\iiint_V \left[\vec{\nabla}\cdot(\rho|v|\,\vec{v}) + (\hat{s}\cdot\vec{\nabla}p) - \rho(\hat{s}\cdot\vec{g})\right]\mathrm{d}V = 0\,.$$

As this integral must equal zero for any arbitrary infinitessimal stream tube, the integrand must equal zero at every point,

$$\vec{\nabla}\cdot(\rho|v|\,\vec{v}) + (\hat{s}\cdot\vec{\nabla}p) - \rho(\hat{s}\cdot\vec{g}) = 0\,.$$

We can apply the product rule to the first term,

$$|v|\,\vec{\nabla}\cdot(\rho\vec{v}) + \rho\vec{v}\cdot\,\vec{\nabla}(|v|) + (\hat{s}\cdot\vec{\nabla}p) - \rho(\hat{s}\cdot\vec{g}) = 0\,.$$

By Equation (3.27), the first term is equal to zero and we find the momentum balance along the stream tube in the direction of the flow,

$$\rho\vec{v}\cdot\,\vec{\nabla}(|v|) + (\hat{s}\cdot\vec{\nabla}p) - \rho(\hat{s}\cdot\vec{g}) = 0\,.$$

The dot product of the velocity with gradient of the magnitude of the velocity will simply give

$$\rho\vec{v}\cdot\vec{\nabla}(|v|) = \rho|v|\frac{\mathrm{d}|v|}{\mathrm{d}\hat{s}}$$

that is, $\rho|v|$ times the derivative of $|v|$ in the $\hat{s}$ direction. The dot product of $\hat{s}$ and $\vec{\nabla}p$ gives the derivative of p in the $\hat{s}$ direction and the dot product of $\hat{s}$ and $\vec{g}$ gives the component of $\vec{g}$ in the $\hat{s}$ direction. This leads to

$$\rho|v|\frac{\mathrm{d}|v|}{\mathrm{d}\hat{s}} + \frac{\mathrm{d}p}{\mathrm{d}\hat{s}} - \rho g_s = 0\,.$$

Looking at a small change in the $\hat{s}$ direction, $\mathrm{d}s$, we see that

$$v\mathrm{d}v + \frac{\mathrm{d}p}{\rho} - g_s\mathrm{d}s = 0\,.$$

I have dropped the absolute value signs on the velocity for clarity. If we assume that $\vec{g} = -g\hat{k}$, we can use the relation that

$$\begin{aligned} g_s\mathrm{d}s &= \vec{g}\cdot\hat{s}\mathrm{d}s \\ &= -g\hat{k}\cdot\hat{s}\mathrm{d}s \\ &= -g\mathrm{d}z\,. \end{aligned}$$

We therefore have

$$v\mathrm{d}v + \frac{\mathrm{d}p}{\rho} + g\mathrm{d}z = 0\,,$$

along a streamline.

If the fluid is incompressible, this can be easily integrated easily to give

$$p + \rho\frac{v^2}{2} + \rho g z = \text{constant.} \tag{3.29}$$

This is the Bernoulli equation for incompressible flow. For compressible flow, one must know the relationship between p and ρ before integrating. If we assume isentropic processes of perfect gases, we also know that

$$\frac{p}{\rho^k} = \text{constant.}$$

This can be used to find the compressible version of Bernoulli's law:

$$\frac{k}{k-1}\frac{p}{\rho} + \frac{v^2}{2} + gz = \text{constant.} \tag{3.30}$$

3.8 Conservation of energy

The Eulerian form of the energy equation is found by chosing η to be the specific energy, e, in Equation (3.20). By Equation (3.4), the time rate of change of energy for a fixed mass (the left hand side) is equal to the rate of energy added through heat transfer, $\dot{Q}$, minus the rate at which work is done on the environment, $\dot{W}$. This gives the relation

$$\dot{Q} - \dot{W} = \frac{\mathrm{d}}{\mathrm{d}t} \iiint_V \rho e \; \mathrm{d}V + \oint\!\!\!\oint_S \rho e \, \vec{v} \cdot \hat{n} \, \mathrm{d}S \,. \tag{3.31}$$

We will separate the work done on the environment into two components, the work done by the pressure force, $\dot{W}_p$, and all other work, $\dot{W}_{\text{other}}$.

The rate of work done on the environment through a differential portion of the control volume surface is the dot product of the pressure force with the velocity here.

$$\mathrm{d}\dot{W}_p = \vec{F}_p \cdot \vec{v} = p\hat{n}\mathrm{d}S \cdot \vec{v}. \tag{3.32}$$

In this case, $\vec{F}_p = p\hat{n}\,\mathrm{d}S$ (without the negative sign) because the force on the environment points out of the control volume, in the same direction as $\hat{n}$.

The total rate of work done by the pressure force is therefore

$$\dot{W}_p = \oint\!\!\!\oint_S p\hat{n} \cdot \vec{v} \, \mathrm{d}S \,. \tag{3.33}$$

Combining this with Equation (3.31), we get

$$\dot{Q} - \dot{W}_{\text{other}} + \oint\!\!\!\oint_S p\hat{n} \cdot \vec{v} \, \mathrm{d}S = \frac{\mathrm{d}}{\mathrm{d}t} \iiint_V \rho e \; \mathrm{d}V + \oint\!\!\!\oint_S \rho e \, \vec{v} \cdot \hat{n} \, \mathrm{d}S \,. \tag{3.34}$$

We will expand the specific energy, e, into its components. It contains the specific internal energy, u, the specific kinetic energy $v^2/2$, and the specific potential energy, gz. This gives

$$\dot{Q} - \dot{W}_{\text{other}} + \oint\!\!\!\oint_S p\hat{n} \cdot \vec{v} \, \mathrm{d}S = \frac{\mathrm{d}}{\mathrm{d}t} \iiint_V \rho \left(u + \frac{v^2}{2} + gz \right) \mathrm{d}V + \oint\!\!\!\oint_S \rho \left(u + \frac{v^2}{2} + gz \right) \vec{v} \cdot \hat{n} \, \mathrm{d}S \,. \tag{3.35}$$

We now combine the surface integrals into one integral

$$\dot{Q} - \dot{W}_{\text{other}} = \frac{\mathrm{d}}{\mathrm{d}t} \iiint_V \rho \left(u + \frac{v^2}{2} + gz \right) \mathrm{d}V + \oint\!\!\!\oint_S \rho \left(u + \frac{p}{\rho} + \frac{v^2}{2} + gz \right) \vec{v} \cdot \hat{n} \, \mathrm{d}S \,. \tag{3.36}$$

This is the Eulerian form of the energy equation that we will use. We notice that the specific enthalpy, $h = u + p/\rho$, naturally appeared in the flux term.

Introduction to Viscosity 4

This chapter serves to provide a *very* brief introduction to viscosity and the differential treatment of fluid flow. It will be useful in the final two chapters of the course.

4.1 Viscosity

The viscosity of a fluid is a measure of how much the fluid resists deformation under the action of shear stresses. Everyone should be familiar with fluids of various viscosities. For example, water has a lower viscosity than honey. To characterize this effect we will summarize a very classical experiment. In this experiment, a fluid is held between two "infinite" solid boundaries. One boundary is stationary while the other moves with a constant velocity, as shown in Figure 4.1.

When one performs this experiment, three things are noted:

1. The velocity of the fluid in contact with the solid wall always is always equal to the velocity of the wall. This is refered to as the *no-slip condition.* It is a very good approximation to reality and will be assumed to be true for this course.

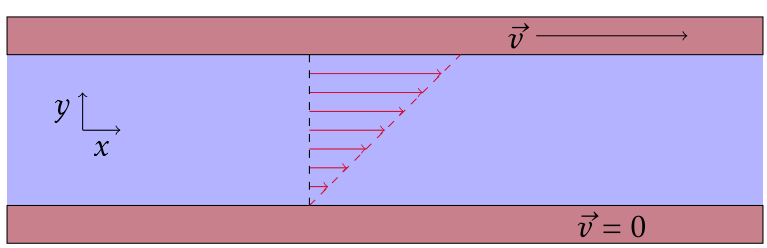


Figure 4.1: Viscous flow between two plates. Bottom plate is stationary and top plate moves with horizontal velocity, $\vec{v}$.

No-slip condition

To a very high degree of physical accuracy, the velocity of a fluid that is in contact with a solid bounday has the same velocity as the boundary.

2. The velocity profile is linear.

3. The force needed to maintain the movement of the upper plate is proportional to the area of the plate and to the slope of the velocity profile.

These observations lead to the relation (valid only for this situation)

$$\frac{\vec{F}}{S} = \tau_{xy} = \mu \frac{\mathrm{d}v_x}{\mathrm{d}y}. \tag{4.1}$$

The constant of proportionality, μ, is the viscosity of the fluid. It is used to relate the shear stress within the fluid to the gradient of the velocity field. It is different for different fluids and normally varies most strongly with the temperature of the fluid.

4.1.1 Fluid stress tensor

Up to now, the only stresses we have seen in a fluid are compressive and caused by the pressure. The stress tensor, $\bar{\sigma}$, for a fluid undergoing only pressure stresses is therefore,

$$\bar{\sigma} = \begin{bmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix}. \tag{4.2}$$

The stresses are negative because, by tradition, tensions are positive. The general stress tensor is

$$\bar{\sigma} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix}. \tag{4.3}$$

The derivation of these terms for a fluid in general is very complicated, in fact it is really beyong a typical undergrad education. I showed some arguments in class about how these stresses should look. In the end, we came up with the expressions:

$$\sigma_{xx} = -p - \frac{2}{3}\mu\vec{\nabla}\cdot\vec{v} + 2\mu\frac{\partial v_x}{\partial x} \tag{4.4}$$

$$\sigma_{yy} = -p - \frac{2}{3}\mu\vec{\nabla}\cdot\vec{v} + 2\mu\frac{\partial v_y}{\partial y} \tag{4.5}$$

$$\sigma_{zz} = -p - \frac{2}{3}\mu\vec{\nabla}\cdot\vec{v} + 2\mu\frac{\partial v_z}{\partial z} \tag{4.6}$$

$$\tau_{xy} = \mu\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) \tag{4.7}$$

$$\tau_{xz} = \mu\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right) \tag{4.8}$$

$$\tau_{yz} = \mu\left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y}\right) \tag{4.9}$$

$$ \tag{4.10}$$

5 Dimensional Analysis

Most practical fluid-mechanics problems are far to complicated to solve exactly on paper. This leaves two options, numerical computation or experiment. This chapter is concerned with experiment. Often, it would be to difficult to conduct experiments on the full scale. It is far preferrable to conduct smaller-scale experiments. For example, we would rather test a scale model of an airplane in a wind tunnel, rather than construct a full-size “practice” aircraft and evaluate its performance in the atmosphere. However, we must be sure that the experiment that we conduct on a reduced scale is actually representative of the real flow. It isn’t enough to simply scale down the geometry.

5.1 The Buckingham Pi theorem

The Pi theorem state that if a situation is a function of n dimensional quantities, it can be re-expressed as a relation of only m *non-dimensional* variables. The reduction, $k = n - m$ is equal to the number of independent dimensions in the problem. In class we saw steps that we can use to find these groups:

1. List the n dimensional variables used in the problem.
2. List the dimensions of each variable
3. Determine the number of independent dimensions present in the problem. This is the reduction, k. It is equal to the maximum number of variable that can be chosen which cannot form a non-dimensional group amoung themselves. If you chose wrong here, the analysis will fail and you can return to this step.
4. Chose k variables that will be used to form the m non-dimensional groups. If possible, take density, velocity, and a length.
5. Form the m non-dimensional groups taking each remaining variable and using the k repeated variables.

5.2 Similarity

We have seen that all physical relations can be re-expressed in a *non-dimensional form.* In doing this, we discussed two types of similarity, geometric similarity and complete similarity. Geometric similarity is necessary, but not sufficient to ensure complete similarity.

Geometric similarity

Two situations are geometrically similar if all shapes are the same. All lengths in two situations are simply scaled by a constant factor.

Complete similarity

Two situations are completely similar if they are geometrically similar *and* the numerical value of all non-dimensional groups in the problem are exactly equal.

For experiments to be conducted at a reduced scale, the situations must be completely similar for meaningful results to be obtained. This is done by ensuring that the numercal value of all relevant Pi groups are exactly the same in both situations.

6 Internal Viscous Flows

In this chapter, we investigate viscous flow in circular pipes. We will be interested in the losses that are expected for particular piping systems containing typical elements (valves, elbows, etc.).

6.1 Steady flow in a pipe

Our first goal will be to understand how a viscous fluid moves through a circular pipe. Experience has shown that, when a flow enters a pipe, after a certain length of the flow, the velocity profile no longer changes through the pipe. This allows for the definition of two regions:

the entry region: The velocity profile changes as the flow moves through the pipe.

fully developed flow: The velocity profile is no longer a function of the distange travelled through the pipe.

We will investigate the velocity profile of a steady flow in a circular pipe. To do this, we will start with some assuptions:

1. The flow is steady. At no point in space does any property vary in time.
2. The flow is axially symmetric. No variable is a function of θ. Therefore, all derivatives in the θ direction are equal to zero.
3. The flow does not have a radial component.
4. The flow is fully developed. The velocity is not a funtion of x, the distance along the pipe.
5. We will ignore gravitational effects for now.

We saw in class that this leads to a velocity profile that is parabolic and a is related to the rate of pressure drop in the pipe as well as the viscosity,

$$v_x(r) = \frac{1}{4\mu}\frac{\mathrm{d}p}{\mathrm{d}x}(r^2 - R^2) \tag{6.1}$$

We can integrate this expression over the cross section of the pipe to find the flow rate,

$$Q = -\frac{\pi R^4}{8\mu}\frac{\mathrm{d}p}{\mathrm{d}x}. \tag{6.2}$$

We can then find the average velocity, $\bar{v}$, in the pipe,

$$\bar{v} = \frac{Q}{A} = \frac{Q}{\pi R^2} = -\frac{R^2}{8\mu}\frac{\mathrm{d}p}{\mathrm{d}x} = -\frac{D^2}{32\mu}\frac{\mathrm{d}p}{\mathrm{d}x}. \tag{6.3}$$

We saw that the pressure in the pipe drops at a constant rate, therefore $\frac{\mathrm{d}p}{\mathrm{d}x} = \frac{\Delta p}{L}$,

$$\bar{v} = -\frac{D^2}{32\mu}\frac{\Delta p}{L}. \tag{6.4}$$

Solving for the pressure loss, Δp, one finds

$$\Delta p = 32\left(\frac{L}{D}\right)\frac{\mu\bar{v}}{D} = 64\left(\frac{L}{D}\right)\left(\frac{\mu}{\rho\bar{v}D}\right)\frac{\rho\bar{v}^2}{2} = \frac{64}{\mathrm{Re}}\left(\frac{L}{D}\right)\frac{\rho\bar{v}^2}{2} \tag{6.5}$$

6.2 Laminar vs turbulent flow

The expressions for the pressure drop in the previous section are very conveninet. Unfortunately, the assuptions on which the derivations are based are almost never true. We saw several videos demonstrating the difference between laminar and turbulent flow. In turbulent flow, properties at any point vary constantly and there are seemingly random components of the velocity field. The transition from laminar to turbulent flow occurs based on the flow Reynolds number, typically at a value near 2300. For turbulent flows, finding exact expressions for the pressure drop is impossible.

Fortunately, dimensional analysis can help. We saw that the non-dimensional pressure drop in a pipe can only be a function of the Reynolds number, the relative roughness, and the ratio of the length to the diameter.

$$\frac{\Delta p}{\frac{1}{2}\rho\bar{v}^2} = g\left(\mathrm{Re}, \frac{\epsilon}{D}, \frac{L}{D}\right). \tag{6.6}$$

Experience shows that the pressure drop is directly proportional to the length, therefore

$$\frac{\Delta p}{\frac{1}{2}\rho\bar{v}^2} = \left(\frac{L}{D}\right) f\left(\mathrm{Re}, \frac{\epsilon}{D}\right). \tag{6.7}$$

We call the function, f, the friction factor. It is only a function of the relative roughness and the Reynolds number. For laminar flow, we see from the previous section that $f = 64/\mathrm{Re}$. However, for turbulent flow, it must be determined experimentally. For us, we can find it using the Moody diagram, Figure 6.1.

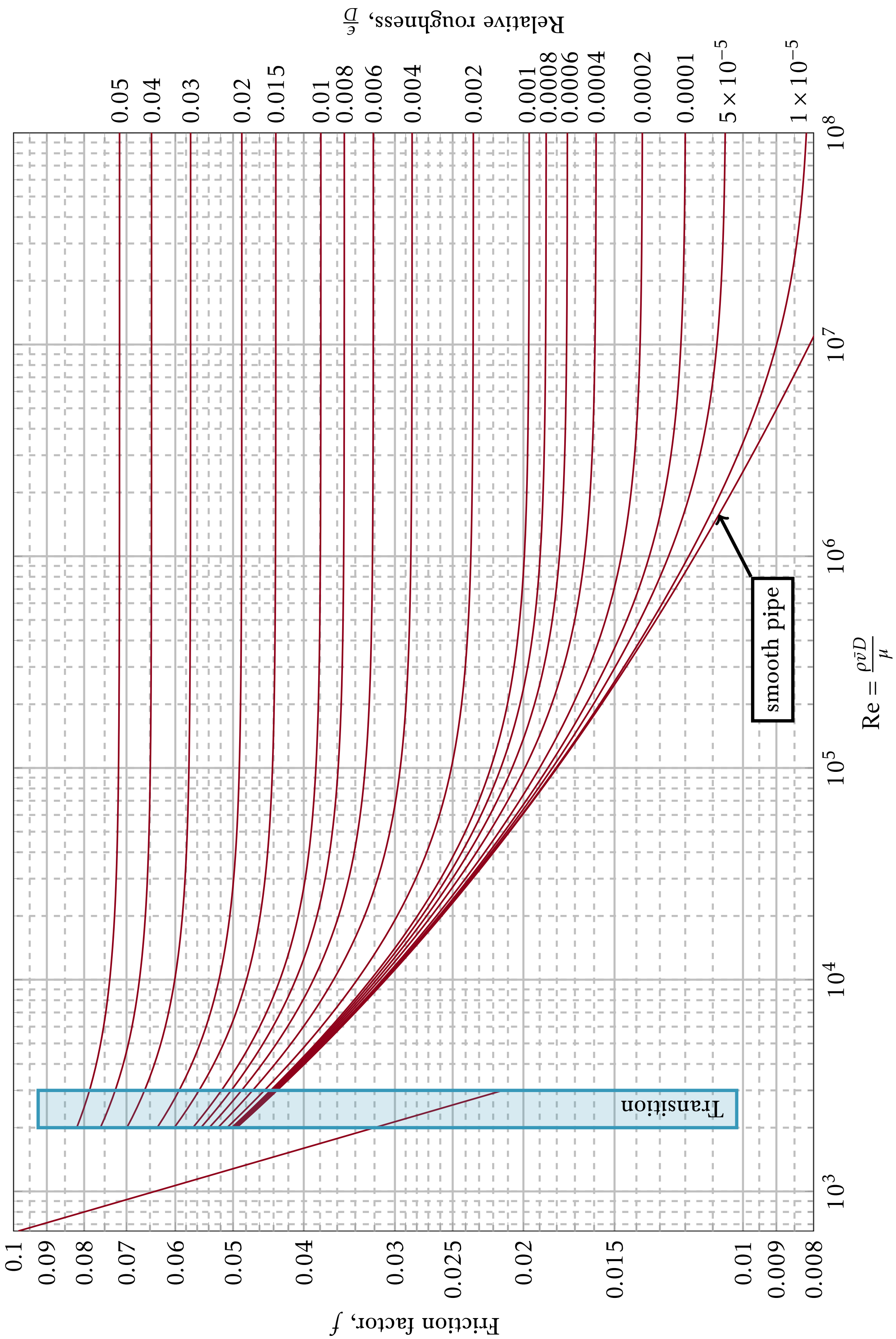


Figure 6.1: Friction factor for fully developed viscous flows in pipes: The Moody diagram, generated through solution of the Colebrook equation.

Table 6.1: Typical roughness of common engineering materials

Material	Roughness, ϵ (mm)
Riveted steel	0.9–9
Concrete	0.3–3
Wood stave	0.2–0.9
Cast iron	0.26
Galvanized iron	0.15
Asphalted cast iron	0.12
Commercial steel or wrought iron	0.046
Drawn tubing	0.0015

6.3 Energy considerations in a pipe flow

If we apply the Eulerian form of conservation of energy, derived in Section 3.8, to a section of a pipe flow in steady state, we can find the energy losses that are caused by viscosity. The energy-balance equation for steady state is

$$\dot{Q} - \dot{W}_{\text{other}} = \oiint_S \rho \left(u + \frac{p}{\rho} + \frac{v^2}{2} + gz \right) \vec{v} \cdot \hat{n} \, \mathrm{d}S \,. \tag{6.8}$$

In a section of a pipe, there is no work done on the environement, therefore $\dot{W}_{\text{other}} = 0$. The pipe also has only one entrance and one exit, therefore

$$\dot{Q} = \iint_{S_{\text{entrance}}} \rho \left(u + \frac{p}{\rho} + \frac{v^2}{2} + gz \right) \vec{v} \cdot \hat{n} \, \mathrm{d}S \;+\; \iint_{S_{\text{exit}}} \rho \left(u + \frac{p}{\rho} + \frac{v^2}{2} + gz \right) \vec{v} \cdot \hat{n} \, \mathrm{d}S \,. \tag{6.9}$$

We will assume that the internal energy, pressure, and height are approximately constant at both the entrance and exit,

$$\begin{aligned} \dot{Q} = -\left(u + \frac{p}{\rho} + gz \right)\Bigg|_{\text{entrance}} \iint_{S_{\text{entrance}}} \rho \vec{v} \cdot \hat{n} \, \mathrm{d}S + \iint_{S_{\text{entrance}}} \rho \frac{v^2}{2} \vec{v} \cdot \hat{n} \, \mathrm{d}S \;+ \\ \left(u + \frac{p}{\rho} + gz \right)\Bigg|_{\text{exit}} \iint_{S_{\text{exit}}} \rho \vec{v} \cdot \hat{n} \, \mathrm{d}S + \iint_{S_{\text{exit}}} \rho \frac{v^2}{2} \vec{v} \cdot \hat{n} \, \mathrm{d}S \, \mathrm{d}S \,. \end{aligned}$$

This can be simplified as

$$\begin{aligned} \dot{Q} = -\dot{m} \left(u + \frac{p}{\rho} + gz \right)\Bigg|_{\text{entrance}} + \iint_{S_{\text{entrance}}} \rho \frac{v^2}{2} \vec{v} \cdot \hat{n} \, \mathrm{d}S \;+ \\ \dot{m} \left(u + \frac{p}{\rho} + gz \right)\Bigg|_{\text{exit}} + \iint_{S_{\text{exit}}} \rho \frac{v^2}{2} \vec{v} \cdot \hat{n} \, \mathrm{d}S \, \mathrm{d}S \,. \end{aligned}$$

Unfortunately, the velocity cannot be considered constant in a pipe flow. If it could, we could remove the $v^2/2$ from the remaining integrals and we would have also have the mass flow rate remaining. In order to accomlish something similar, we define a ***kinetic energy coefficient***, α,

$$\iint_S \rho \frac{v^2}{2} \vec{v} \cdot \hat{n} \, \mathrm{d}S = \alpha \frac{\bar{v}^2}{2} \iint_S \rho \vec{v} \cdot \hat{n} \, \mathrm{d}S \,. \tag{6.10}$$

For laminar flow, we can evaluate the integrals exactly and find $\alpha = 2$. For a turbulent flow, mixing causes the velocity profile to be much closer to contant, we generally assume $\alpha = 1$ for turbulent flow.

This allows us to simplify our application of the energy equation to pipe flows to

$$\dot{Q} = -\dot{m}\left(u + \frac{p}{\rho} + \alpha\frac{\bar{v}}{2} + gz\right)\Bigg|_{\text{entrance}} + \dot{m}\left(u + \frac{p}{\rho} + \alpha\frac{\bar{v}}{2} + gz\right)\Bigg|_{\text{exit}} . \tag{6.11}$$

This can be re-ordered to give

$$\left(\frac{p}{\rho} + \alpha\frac{\bar{v}}{2} + gz\right)\Bigg|_{\text{entrance}} - \left(\frac{p}{\rho} + \alpha\frac{\bar{v}}{2} + gz\right)\Bigg|_{\text{exit}} = -\frac{\dot{Q}}{\dot{m}} + u_{\text{exit}} - u_{\text{entrance}} . \tag{6.12}$$

We call the term, $\left(\frac{p}{\rho} + \alpha\frac{\bar{v}}{2} + gz\right)$, mechanical energy. It is the energy that can be used to do useful work. The terms on the right-hand side are the losses of mechanical energy to heat. We use the symbol h_L for the losses,

$$\left(\frac{p}{\rho} + \alpha\frac{\bar{v}}{2} + gz\right)\Bigg|_{\text{entrance}} - \left(\frac{p}{\rho} + \alpha\frac{\bar{v}}{2} + gz\right)\Bigg|_{\text{exit}} = h_L . \tag{6.13}$$

6.3.1 Losses

There are two types of losses to consider, major losses caused by viscosity in straight sections of pipe, and minor losses caused by fittings (valves, elbows, etc.). We have already found a way to get the pressure losses in a horizontal pipe of constant diameter,

$$\frac{\Delta p}{\frac{1}{2}\rho\bar{v}^2} = \left(\frac{L}{D}\right) f . \tag{6.14}$$

The loss can be written as

$$h_M = \frac{\Delta p}{\rho} = f\left(\frac{L}{D}\right)\left(\frac{\bar{v}^2}{2}\right). \tag{6.15}$$

Minor losses can be calculated in two ways, either as a function of a loss coefficient, K,

$$h_m = K\frac{\bar{v}^2}{2} , \tag{6.16}$$

or through the definition of an equivalent length, $(L/D)_E$,

$$h_m = f\left(\frac{L}{D}\right)_E \frac{\bar{v}^2}{2} . \tag{6.17}$$

In both cases, K or $(L/D)_E$, must be looked up in tables or graphs. The total losses are the sum of the major and minor losses,

$$\left.\left(\frac{p}{\rho}+\alpha\frac{\bar{v}}{2}+gz\right)\right|_{\text{entrance}}-\left.\left(\frac{p}{\rho}+\alpha\frac{\bar{v}}{2}+gz\right)\right|_{\text{exit}}=\sum f\left(\frac{L}{D}\right)\left(\frac{\bar{v}^2}{2}\right)+\sum K\frac{\bar{v}^2}{2}+\sum f\left(\frac{L}{D}\right)_E\frac{\bar{v}^2}{2}. \quad (6.18)$$

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[2] G. N. Patterson. *Introduction to the Kinetic Theory of Gas Flows.* University of Toronto Press, Toronto, 1971.

[3] A. Reynolds. *Papers on Mechanical and Physical Subjects: The sub-mechanics of the universe*, volume 3. Cambridge University Press, 1903.

Useful Mathematical Relations

A

This appendix contains mathematical relations that are important throughout this course. Rather than rigorous proofs of each relation, the goal is to argue their truth based more on intuition. Hopefully, this intuitive approach will make you comfortable with the validity of the expressions and give you a firm physical feeling for what each relation *means* physically.

A.1 Vector fields

A vector field is a mathematical field that assigns a vector to each point in a space. For us, the space will usually be physical space and the vector will often be the fluid velocity. This special case is known as a *velocity field*. It describes the velocity of the fluid at every point of a flow. For example, Figure A.1 shows a graphical representation of the two-dimensional vector field

$$\vec{v}(x,y) = v_x(x,y)\,\hat{\imath} + v_y(x,y)\,\hat{\jmath} = \sin y\,\hat{\imath} + \cos x\,\hat{\jmath}. \tag{A.1}$$

At each (x,y) point in space, there is a corresponding vector, $(v_x, v_y) = (\sin y,\ \cos x)$. The arrows indicate the magnitude and direction of the vector that exists at point marked by a black circle at the tail of the arrow.

A.2 The mean-value theorem

The mean-value theorem states that a function, f, that is continuous and differentiable on an interval $a \le x \le b$ will have a derivative that is equal to $\frac{f(b)-f(a)}{b-a}$ at some point $x^\star$, where $a \le x^\star \le b$. Another way of saying this is that

$$f'(x^\star) = \frac{f(b)-f(a)}{b-a}, \tag{A.2}$$

where f' is the derivative of f with respect to x and $x^\star$ is some value of x that is between a and b. This is shown graphically in Figure A.2. As can be observed in this figure, there is always

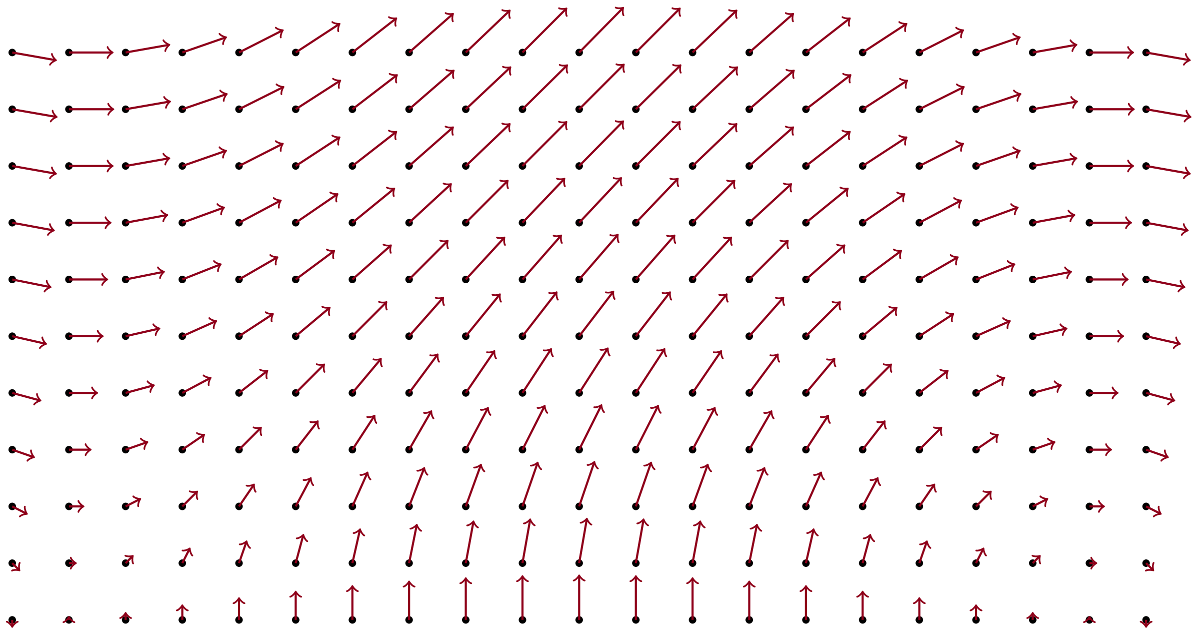

Figure A.1: Example two-dimensional vector field.

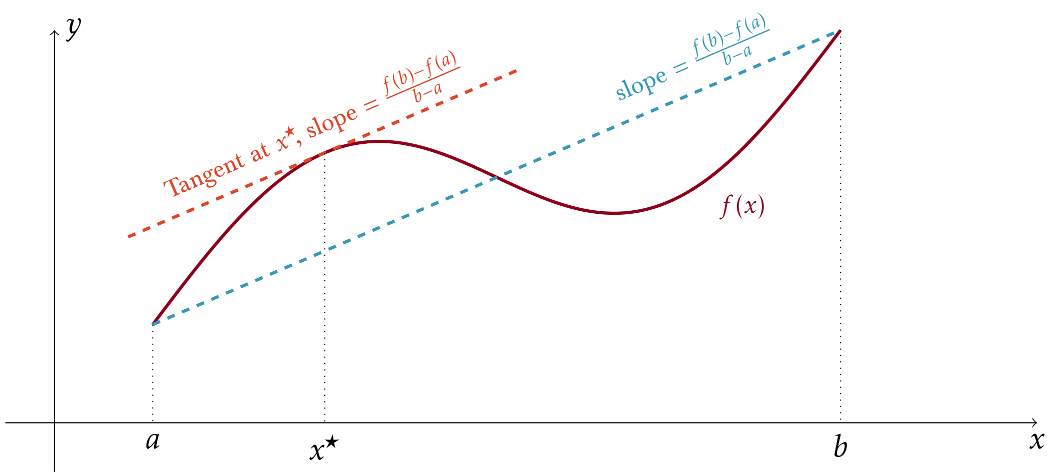

Figure A.2: A graphical representation of the mean-value theorem.

at least one point for which the derivative of f is equal to the slope of the line connecting the two endpoints. There may also be multiple such points. A proof of this concept is not provided here, however, it is hoped that the truth of this theorem seems obvious after some thought.

A.2.1 The mean-value theorem for definite integrals

A useful form of the mean-value theorem applies to definite integrals. We can obtain this form by defining f to be the definite integral of another continuous differentiable function, g, over the range from a to x,

$$f(x) = \int_a^x g(\xi)\,\mathrm{d}\xi = G(x) - G(a).$$

Here, by the fundamental theorem of calculus, G is the anti-derivative of g. Now, the derivative of f with respect to x is given by

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\Big(G(x) - G(a)\Big) = \frac{\mathrm{d}G(x)}{\mathrm{d}x} - 0 = g(x), \tag{A.3}$$

as the derivative of $G(x)$ is $g(x)$ and the derivative of $G(a)$ is zero. Now, by the application of Equation (A.2) to the function, $f(x)$ using the interval $a \le x \le b$, we have

$$g(x^\star) = \underbrace{\boxed{f'(x^\star) = \frac{f(b) - f(a)}{b - a}}}_{\text{Equation (A.2)}} = \frac{\int_a^b g(\xi)\,\mathrm{d}\xi - \int_a^a g(\xi)\,\mathrm{d}\xi}{b - a} = \frac{\int_a^b g(\xi)\,\mathrm{d}\xi}{b - a},$$

as $\int_a^a g(\xi)\,\mathrm{d}\xi = 0$. This can be rearranged to give

$$g(x^\star)\,(b - a) = \int_a^b g(\xi)\,\mathrm{d}\xi\,. \tag{A.4}$$

Here, $x^\star$ is some value of x between a and b. This should be interpreted as saying that the continuous function, g, must pass through its average value on the interval $a \le x \le b$ at least once within this interval. Of course, it could happen that g passes the average value multiple times within the interval. This relation is demonstrated graphically in Figure A.3. If one multiplies the average value of the function, $g(x^\star)$ by the size of the interval $(b - a)$, one would find the result of the integral. It should seem obvious that a continuous function must pass through its average value at least once within such an interval.

A.3 Integration on curved surfaces

In this course, we often need to integrate function on curved surfaces. The functions the we integrate are often involve the unit normal to the surface, $\hat{n}$. For example, the net force of a

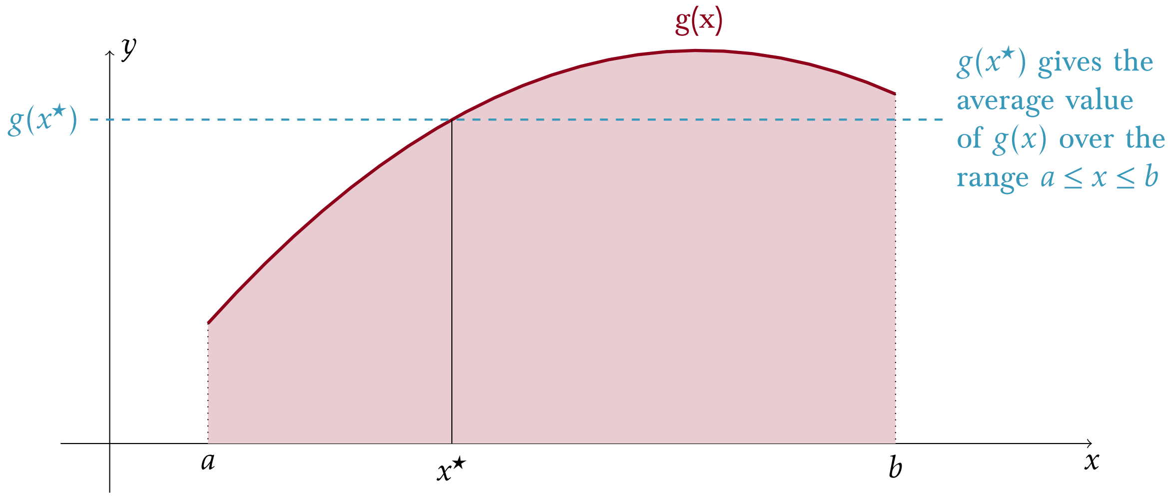

Figure A.3: A graphical representation of the mean-value theorem. The continuous function must pass through its average value within any interval.

surface caused by fluid pressure is

$$\vec{F}_p = \iint_S -p\,\hat{n}\,\mathrm{d}S\,. \tag{A.5}$$

One technique that can be used to evaluate such integrals on curved surfaces is to define a parameterization (or mapping) of the surface. This is done by defining a vector function of two variables (often u and v),

$$\vec{r}(u,v) = r_x(u,v)\,\hat{\imath} \;+\; r_y(u,v)\,\hat{k} \;+\; r_z(u,v)\,\hat{k}\,. \tag{A.6}$$

This function is defined such that, for all values between appropriate limits ($u_min < u < u_max$ and $v_min < v < v_max$), $\vec{r}$ gives a point on the curved surface and each point on the three dimensional surface corresponds to exactly one pair of u and v.

Differential changes in u and v will cause a differential change in $\vec{r}$ given by

$$\mathrm{d}\vec{r} = \frac{\partial\vec{r}}{\partial u}\mathrm{d}u \;+\; \frac{\partial\vec{r}}{\partial v}\mathrm{d}v\,. \tag{A.7}$$

If we zoom in near a point with $u = u_0$ and $v = v_0$, the differential area, $\mathrm{d}S$, will be the area enclosed between the lines of constant u with $u = u_0$ and $u = u_0 + \mathrm{d}u$ as well as the lines of constant v with $v = v_0$ and $v = v_0 + \mathrm{d}v$, as shown in Figure A.4.

The area of this parallelogram can be computed by defining the vectors

$$\mathrm{d}\vec{r}_1 = \vec{r}(u_0 + \mathrm{d}u, v_0) - \vec{r}(u_0, v_0) \tag{A.8}$$

$$\mathrm{d}\vec{r}_2 = \vec{r}(u_0, v_0 + \mathrm{d}v) - \vec{r}(u_0, v_0)\,. \tag{A.9}$$

These vector can be computed, using Equation (A.7), as

$$\mathrm{d}\vec{r}_1 = \frac{\partial\vec{r}}{\partial u}\mathrm{d}u \tag{A.10}$$

$$\mathrm{d}\vec{r}_2 = \frac{\partial\vec{r}}{\partial v}\mathrm{d}v\,. \tag{A.11}$$

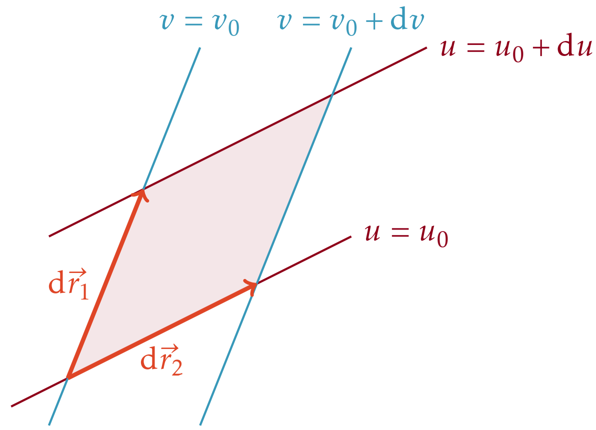

Figure A.4: The differential area, $\mathrm{d}S$.

The area of the parallelogram is given by the magnitude of the cross product of these vectors,

$$\begin{aligned} \mathrm{d}S &= \left|\frac{\partial\vec{r}}{\partial u}\mathrm{d}u \times \frac{\partial\vec{r}}{\partial v}\mathrm{d}v\right| \\ &= \left|\frac{\partial\vec{r}}{\partial u} \times \frac{\partial\vec{r}}{\partial v}\right| \mathrm{d}u\,\mathrm{d}v\,. \end{aligned} \tag{A.12}$$

The vectors in Equations (A.10) and (A.11) can also be used to find $\hat{n}$. We know that the cross product of any two non parallel vectors will give a third vector that is normal to both original vectors. This vector can be given unit length by dividing by its magnitude,

$$\begin{aligned} \hat{n} &= \frac{\frac{\partial\vec{r}}{\partial u} \times \frac{\partial\vec{r}}{\partial v}\,\mathrm{d}u\,\mathrm{d}v}{\left|\frac{\partial\vec{r}}{\partial u} \times \frac{\partial\vec{r}}{\partial v}\right|\mathrm{d}u\,\mathrm{d}v} \\ &= \frac{\frac{\partial\vec{r}}{\partial u} \times \frac{\partial\vec{r}}{\partial v}}{\left|\frac{\partial\vec{r}}{\partial u} \times \frac{\partial\vec{r}}{\partial v}\right|} \end{aligned} \tag{A.13}$$

This vector will be normal to the surface on of unit length, but, when defining the parameterization, its not always easy to predict to which side of the surface it will point. If it turns out to point in the wrong direction, it can be flipped by multiplying by negative one.

A.4 Flux

For this course, we will often need to compute the rate at which something flows through some surface. To do this, we define the mathematical concept of a flux. If a vector field describes the velocity of a medium, the flux gives the rate at which volume of that medium moves through a surface. Let's look at how this can be computed. We will start with a case that is relatively simple and then generalize until we find the most general expression.

A.4.1 Flux when the velocity field is constant in space and normal to a planar surface, S.

Figure A.5a shows a velocity filed that is constant in space (but still a function of time) and normal to a surface, S. Take $\hat{n}$ to be the unit normal to the surface. We imagine that at some time, t_1, there is another surface, S_2, that is exactly equal in shape and size to S that lies exactly on top of S. The difference is, while S is fixed in space, our new surface is glued to the moving fluid. At a time in the future, $t + \Delta t$, it will have moved a distance in the direction of the fluid velocity. This is shown in Figure A.5b. The volume of fluid that passes through surface, S, during Δt in the direction of the unit normal is the volume between S and S_2. This volume is the surface area of the base, S, times the length, L,

$$V = SL \tag{A.14}$$

The length, L, is the distance that the fluid moved between t and $t + \Delta t$. This is

$$L = \int_t^{t+\Delta t} v \, \mathrm{d}t, \tag{A.15}$$

which gives a volume of

$$V = S \int_t^{t+\Delta t} v \, \mathrm{d}t. \tag{A.16}$$

When we speak of the flux, we are talking about a rate. For example, how many litres per second are passing through the surface at some time. This is the derivative of Equation A.16 with respect to time. By definition, this is

$$\begin{aligned} \frac{\mathrm{d}V}{\mathrm{d}t} &= \lim_{\Delta t \to 0} \frac{V(t+\Delta t) - V(t)}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{S \int_t^{t+\Delta t} v \, \mathrm{d}t - S \int_t^{t} v \, \mathrm{d}t}{\Delta t} = \lim_{\Delta t \to 0} \frac{S \int_t^{t+\Delta t} v \, \mathrm{d}t - 0}{\Delta t}. \end{aligned} \tag{A.17}$$

The second integral is equal to zero because no fluid passes through the surface between time t and t. We can replace the first integral using the mean-value theorem for definite integrals shown in Section A.2.1,

$$\begin{aligned} \frac{\mathrm{d}V}{\mathrm{d}t} &= \lim_{\Delta t \to 0} \frac{S \, (t + \Delta t - t) \, v(t^\star)}{\Delta t} \\ &= \lim_{\Delta t \to 0} \; S \, v(t^\star). \end{aligned} \tag{A.18}$$

The volume flux is equal to limit as the time interval, Δt becomes infinitely small of the surface area, S, times the average velocity over the time interval, $v(t^\star)$. Since the mean-value theorem

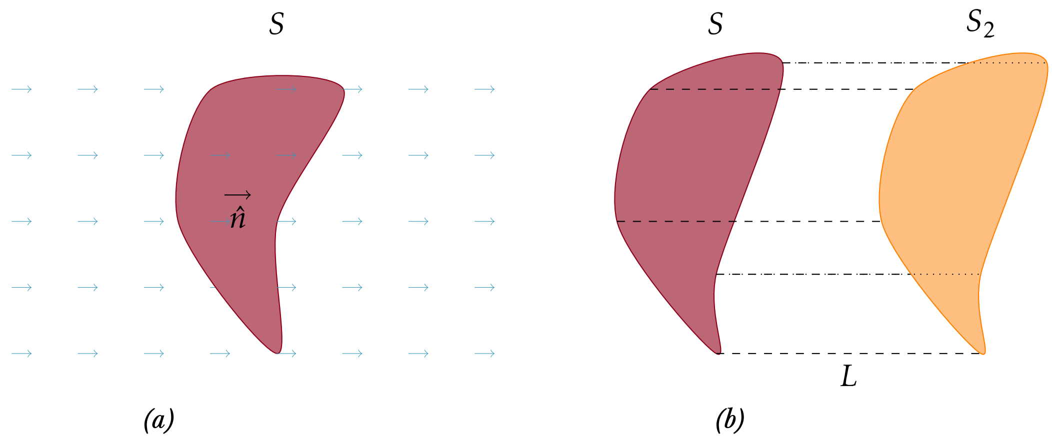

Figure A.5: Illustration of the flux of a velocity field through a surface to which it is normal.

tells us that $t^\star$ is always between t and $t + \Delta t$, when Δt becomes zero, both $t + \Delta t$ and $t^\star$ will equal t. Thus, the volumetric flux of a fluid that is flowing normal to a surface is

$$\frac{\mathrm{d}V}{\mathrm{d}t} = Sv\,, \tag{A.19}$$

the surface area times the magnitude of the velocity.

A.4.2 Flux when the velocity field is constant in space but not normal to a planar surface, S.

Now, let's add one more level of complication by removing one restriction from Section A.4.1. We will still take the vector field, $\vec{v}$, to be constant in space, but we will now allow for it to not be normal to the surface. In this case, the volume swept out would be at an angle. This is shown in Figure A.6. The volume swept out in Figure A.6b is now at an angle. The volume of such a shape is

$$V = SH\,. \tag{A.20}$$

In this case, H, is the projection of L on the surface's unit normal. The displacement of the second surface (carried with the moving fluid) is again the integral of its velocity from $t1$ to $t1 + \Delta t$. The volume is therefore,

$$\begin{aligned} V &= \hat{n} \cdot \int\limits_{t1}^{t1+\delta t} \vec{v}\mathrm{d}t \\ &= \int\limits_{t1}^{t1+\delta t} \vec{v} \cdot \vec{n}\mathrm{d}t\,. \end{aligned} \tag{A.21}$$

Following the same logic used in Section A.4.1, the instantaneous volume flux can be found as

$$\frac{\mathrm{d}V}{\mathrm{d}t} = S\vec{v} \cdot \hat{n}\,. \tag{A.22}$$

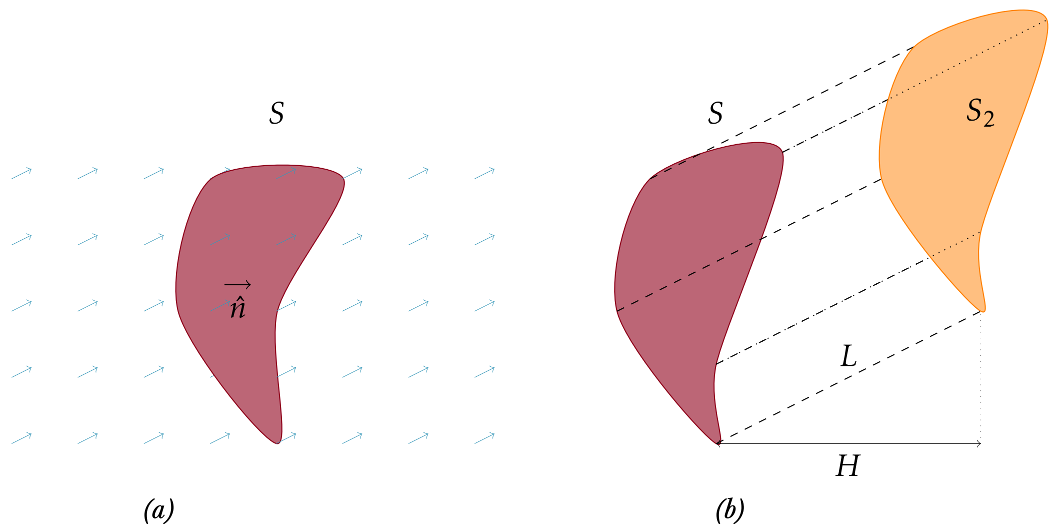

Figure A.6: Illustration of the flux of a velocity field through a surface to which it is not perpendicular.

One way to look at this is to see that it is only the component of $\vec{v}$ that is normal to the surface that contributes to the flux.

A.4.3 Flux when the velocity field is a function of space and time, and the surface is curved.

Finally, to find the flux in its most general form, we consider a vector field, $\vec{v}$, that is a function of position and time as well as a surface that is curved. In order to determine the total flux of such a vector field through such a surface we can imagine breaking the surface up into very small subsurfaces of area $\mathrm{d}S$. On each infinitesimal subsurface, we can take both $\hat{n}$ and $\vec{v}$ to be constant. The total flux through to total surface would then be the sum of the contribution from each tiny subsurface. This is obviously an integral over the surface.

Flux of $\vec{v}$ through a curved surface

The total flux of a vector field, $\vec{v}$, through a curved surface, S is

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \iint_S \vec{v} \cdot \hat{n}\, \mathrm{d}S\,. \tag{A.23}$$

A.5 The divergence of a vector field

Once one is comfortable with the idea of a flux, as described in Section A.4, this can be used to define several important properties of vector fields. One such property that is very important in this course is the divergence. Imagine an arbitrary volume, V, with a closed surface, S, Imagine also a unit normal vector n that is defined on the surface and always points outward. Now consider the meaning of the expression

$$\oiint_S \vec{v} \cdot \hat{n} \, \mathrm{d}S \,. \tag{A.24}$$

This integral over the entire closed surface is the net flux of the vector field out of the volume. When our vector field is the fluid velocity, this is the net volumetric flow rate of fluid out of the volume. Another way of putting this is that it is the rate at which fluid is ***diverging*** from the volume. The definition of the divergence of a vector field at a point is the value of Equation A.24 in the limit that the volume shrinks to a point, normalized by the size of the volume.

Divergence of a vector field

The divergence of a vector field is the net flux of the vector field out of a volume in the limit that the volume shrinks to zero, normalized by the volume,

$$\mathrm{div}\, \vec{v} = \lim_{V \to 0} \frac{1}{V} \oiint_S \vec{v} \cdot \hat{n} \, \mathrm{d}S \,. \tag{A.25}$$

In Cartesian coordinates, this operation takes the form

$$\begin{aligned}
\mathrm{div}\, \vec{v} &= \vec{\nabla} \cdot \vec{v} \\
&= \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) \cdot \left(w_x \hat{i} + w_y \hat{j} + w_z \hat{k} \right) \\
&= \left(\frac{\partial w_x}{\partial x} + \frac{\partial w_y}{\partial y} + \frac{\partial w_z}{\partial z} \right) .
\end{aligned} \tag{A.26}$$

A.6 The divergence theorem

Explain divergence theorem here.

Divergence theorem

$$\iiint_V \vec{\nabla} \cdot \vec{v} \, \mathrm{d}V = \oiint_S \vec{v} \cdot \hat{n} \, \mathrm{d}S \tag{A.27}$$

A.7 The gradient theorem

A useful form of the divergence theorem can be obtained by defining the vector field to be the product of a scalar field and an arbitrary, constant, non-zero vector,

$$\vec{v}(x,y,z) = \vec{a}\ \phi(x,y,z)\,.$$

This can be substituted into Equation (A.27),

$$\iiint_V \vec{\nabla}\cdot(\vec{a}\phi)\ \mathrm{d}V = \oiint_S (\vec{a}\phi)\cdot\hat{n}\ \mathrm{d}S\,. \tag{A.28}$$

The constant vector, $\vec{a}$, can be removed from the derivatives and integral, giving

$$\vec{a}\cdot\iiint_V \vec{\nabla}\phi\ \mathrm{d}V = \vec{a}\cdot\oiint_S \phi\ \hat{n}\ \mathrm{d}S\,. \tag{A.29}$$

Since the vector is arbitrary and non-zero, it can be cancelled from both sides. The result is often referred to as the gradient theorem.

gradient theorem

$$\iiint_V \vec{\nabla}\phi\ \mathrm{d}V = \oiint_S \phi\ \hat{n}\ \mathrm{d}S \tag{A.30}$$

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